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DEAR SIR,

We beg to present you with the first number of our little Scientific Journal and request that you will examine it, with a view to assisting us in filling its pages with subject matter that is worthy of the effort which we have in view.

We desire to publish neat solution of problems in Pure and Applied Mathematics, short cuts to engineering and physical formulæ, theoretical investigation of subjects that are new or interesting from the novelty of treatment, chemical investigations and analyses which are of interest to the profession.

We ask your leniency of criticism for this our first effort, dependent as it is entirely upon home talent and mechanical execution. We believe that in future numbers much of the crudity of the latter will disappear, and that with your contributions the quality of the matter printed will be much improved.

Respectfully,

THE EDITORS.

PROSPECTUS.

"* * * * *Men the workers, ever reaping something new;*
That which they have done but earnest of the things that they shall do:"

Upon the first appearance of a scientific periodical it is well to declare its intent and purpose, and to define the place in scientific literature which it hopes to take; for the obligation is thus incurred to accomplish the purpose, to fulfill the design, and to reach the place. The danger before the untried publication is, as with the individual, that of placing the objective point too high, and in the effort passing ignominiously beneath the goal. Fortunately scientific work is honest, and its creed is truth; and in honest work and honest failure there can be no disgrace.

There is a considerable amount of scientific investigation going on at the present day which is of no contemptible order of merit, and which really deserves the name of research. This class of work is being done by the younger men, who are working earnestly and honestly; for the love of their subject, and for the love of the work itself. These are the men upon whose shoulders, in the coming years, as the masters pass away their mantles fall. These are the men from whose ranks in maturer time, will be called those who are to inherit the mighty responsibility of developing scientific truth, to grope in the darkness further on, to feel the way to clearer thought, to show the way to higher reason, to throw the light upon the mists of doubt and clear away the error.

It is for this class of younger workers that this publication has its being, it is for them to give it success or let it fail. The selection of its name is not from pedantry, but because it has in its sense a twofold meaning. Its pages are open to work in Mathematics, Pure and Applied; to Physics and to Chemistry, the exact sciences of the Bachelor's degree. On the other hand the name suggests the status of the workers, for it is not intended to be an undergraduate's plaything, nor yet does it aspire to the more powerful thought of the master workman.

Its object is to encourage and foster the spirit of investigation and original thought. Its pages are to show research and originality in results which are new. The propounding and solution of questions, which by their novelty are interesting, is also invited.

Mechanically, every effort will be made to secure good work in material and typography, to avoid errors of printing as nearly as may be, and to present an attractive and well proportioned pamphlet.

The number of pages in each issue is not necessarily limited, but will depend entirely upon the amount of matter presented and accepted for publication.

A nominal subscription fee of one dollar per volume of four quarterly numbers is asked, believing it to be worthless unless of at least that pecuniary value. However, on written request, the name of any scientific worker will be placed on the subscription list for the first volume; provided such a request be assumed as an obligation to pay the fee when the volume has been received, if in the recipient's estimation the character of the work in the publication deserves it. Duplicates and reprints of all articles printed will be furnished the authors free of charge.

We hope and we also believe that the masters in scientific thought will not ignore us altogether; but will encourage us from time to time by appearing in our modest pages, for the purpose of encouraging and suggesting lines of thought to younger men, whose earnestness and whose honest work commands respect, albeit they fall into error in their efforts and need correction: for younger hands must yet take up the torch and spread the light which older minds are now preparing.

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THE RAILWAY TRANSITION CURVE.

BY PROF. W. H. ECHOLS.

1. In the *Engineering News* of date May 4, 1889, the writer presented the subject of transition curves in the solution of the following problems.

1. To pass from the tangents to a D° *curve by* n *stations of a* d° *curve.*

2. To pass from the tangents to a D° *curve by* n *stations of a* $d°_1$, *and* m *stations of a* $d°_2$ *curve.*

The general solutions of these problems are there given and the results applied to two particular cases respectively as follows:

(*1.*) When $d° = 2°$ and $n = 2$, we have for the tangent distance

$$T = \left(\frac{5716}{D°} + 7\right) \text{ta}\,\tfrac{1}{2}I° + 200\,\frac{D° - 2}{D°}.$$

The length of the curve is

$$L = \frac{I°}{D°} + 4\,\frac{D° - 2}{D°}.$$

The effect of the transition being to move the curve *in* an amount

$$\left(\frac{21}{3} - \frac{14}{D}\right) \text{se}\,\tfrac{1}{2}I°.$$

In which formulae, $I°$ is the angle of intersection of the tangents in degrees.

(*2.*) When $n=m=d_1=1$ and $d_2=3$, we have the corresponding formulae:

$$T=\left(\frac{5716}{D}+5.2\right)\text{ta}\tfrac{1}{2}I+200\frac{D-2}{D},$$

$$L=\frac{I}{D}+4\frac{D-2}{D},$$

and

$$\left(\frac{12}{4}-\frac{14}{D}\right)\text{se}\tfrac{1}{2}I,$$

2. These formulae being almost identical and quite simple it was proposed to use them in locating railway curves. It was assumed that up to $D=2^\circ$ no easement was necessary in the transition from the tangent to the curve, and it was desgned that for $2^\circ < D \leqq 3$, the curve in (*1*) was to be used while for $3^\circ < D < 6^\circ$ or more, the curve (*2.*) was to be put in.

In view of the possibility of these two particular cases not being "calculated to fulfill all the requirements for a system of transition curves, nor to replace such a system;" it is proposed in the present paper to develope the transition curve in all its generality and to apply the results to deduce a system of transition curves for railway location.

3. The transition curve must afford an easy change of curvature from the tangent to the curve. It is also much to be desired that it should be located at once with the transit by deflection angles and as easily as can be done a simple circular curve. The point of curve, or what is equivalent thereof, the tangent distance must be obtained with about the same ease as that of a simple curve and finally the location must differ as little as may be from that which it would occupy should no transition curve be employed.

These requirements lead us then to the following:

Definition: A Transition Curve is one whose curvature increases per unit of arc in arithmetical progression, or whose change of curvature per unit arc is constant.

The ordinary geometrical equation of such a curve cannot be written because the curvature lacks continuity; it is therefore not a fit subject for infinitesimal analysis but for that of finite differences. From the definition of *curvature* it follows that a transition curve is a succession of circular arcs of unit or of constant length, which subtend central angles in arithmetical progression. It is as such, that the transition curve will be investigated, its functions determined and the final results presented in working shape for the locating engineer.

4. Consider two intersecting straights, meeting at the angle I, which are to be united by a circular arc of radius R, but in order to avoid the sudden change of curvation in passing from the tangent to the arc of circle R, a compound curve of n equal circular arcs of radii, $r_1, r_2 \ldots r_n$ proportional respectively to $1, 1/2, \ldots 1/n$ is to be inserted between the tangent and the given arc.

The functions of the compound curve to be determined are:

(*1.*) The tangent distance T, or the distance from $P.\ C.$ or $P.\ T.$ the point of tangency of the compound curve from I the intersection of the tangents.

(*2.*) The length of the curve of radius R.

(*3.*) The deflections from the tangent at $P.\ C.$ necessary to set the ends of the equal arcs up to the initial point of the R curve; also the deflections from the tangent at the terminal point of the R curve necessary to set the ends of the equal arcs from that point to $P.\ T.$

5. Computation of the Tangent:—

The radii $r_1, r_3, \ldots r_n$ of the equal arcs are so great with respect to the lengths of the arcs that we shall assume that the chords of these arcs are equal to the arcs and therefore to each other.

Let the arcs of radii $r_1, r_2, \ldots r_n$ subtend at the centers O_1, $O_2, \ldots O_n$, of their circles the angles $d_1, d_2, \ldots d_n$, respectively. From each center, as for example, O_3, draw a normal (Fig. 1.) to the initial tangent to meet the arc whose center is O_3, and radius r_3, in a point a_3; and at a_3 draw the tangent to the arc || to the initial tangent to meet the perpendicular to the initial tangent from the center O_3 in a point b_3. Indicate the distances $a_3\, b_3$ and $b_3\, a_4$ by x_3 and z_3 respectively. Through

b_3 and a_4 draw || straights to meet the initial tangent making with it the angle $90° - \frac{1}{2}I$, thus projecting z_3 on the initial tangent into a distance $y_3 = z_3$ ta $\frac{1}{2}I$.

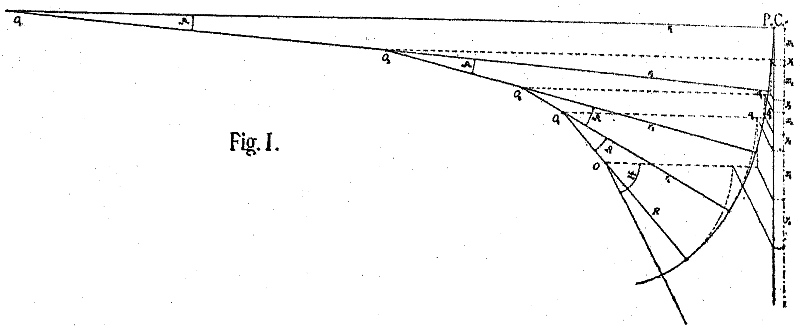


Fig. I.

Beginning at *P. C.* and making the construction as above indicated, we have for the tangent distance of the compound curve, observing that

$$z_m = y_m \text{ ta } \tfrac{1}{2}I.$$

$$T = \begin{cases} \text{R ta } \frac{1}{2}I \\ + x_1 + x_2 + \dots + x_n \\ + y_1 + y_2 + \dots + y_n, \end{cases}$$

$$= R \text{ ta } \tfrac{1}{2}I + \sum_1^n x + \sum_1^n y,$$

$$= (R + \sum_1^n z) \text{ ta } \tfrac{1}{2}I + \sum_1^n x.$$

From the figure

$$\sum_1^n x = \begin{cases} \quad x_1 = (r_1 - r_2) \text{ si } d_1 \\ + x_2 = (r_2 - r_3) \text{ si } (d_1 + d_2) \\ + \dots \dots \\ + x_n = (r_n - R) \text{ si } (d_1 + d_2 + \dots + d_n). \end{cases}$$

also

$$\sum_1^n z = \left\{ \begin{aligned} & z_1 = r_1(1 - \text{co } d_1) - r_2(1 - \text{co } d_1) \\ & \quad = r_1 - r_2 - (r_1 - r_2) \text{ co } d_1 \\ & + z_2 = r_2 - r_3 - (r_2 - r_3) \text{ co } (d_1 + d_2) \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & + z_n = r_n - R - (r_n - R) \text{ co } (d_1 + d_2 + \ldots + d_n) \end{aligned} \right.$$

or

$$\sum_1^n z = r_1 - R - \left\{ \begin{aligned} & (r_1 - r_2) \text{ co } d_1 \\ & + (r_2 - r_3) \text{ co } (d_1 + d_2) \\ & \quad . \quad . \quad . \quad . \quad . \\ & + (r_n - R) \text{ co } (d_1 + d_2 + \ldots + d_n). \end{aligned} \right.$$

put. Cos = 1 − 2 Sin² ½ and reduce.

Since

$$r_1 = 2r_2 = 3r_3 = \ldots = nr_n$$

and

$$d_1 = \tfrac{1}{2}d_2 = \tfrac{1}{3}d_3 = \ldots = \tfrac{1}{n}d_n,$$

we have, making these substitutions

$$\sum_1^n x = \tfrac{1}{2}r_1 \left\{ \begin{aligned} & \text{si } d_1 \\ & + \tfrac{1}{3} \text{ si } 3d_1 \\ & \quad . \quad . \quad . \quad . \quad . \\ & + \frac{2}{n(n-1)} \text{ si } \tfrac{1}{2}n(n-1)d_1 \\ & + \left(\frac{1}{n} - \frac{R}{r_1}\right) \text{ si } \tfrac{1}{2}n(n+1)d_1 \end{aligned} \right.$$

and

$$\sum_1^n z = r_1 - R - \tfrac{1}{2}r_1 \left\{ \begin{aligned} & \text{co } d_1 \\ & + \tfrac{1}{3} \text{ co } 3d_1 \\ & \quad . \quad . \quad . \quad . \quad . \\ & + \frac{2}{n(n-1)} \text{ co } \tfrac{1}{2}n(n-1)d_1 \\ & + \left(\frac{1}{n} - \frac{1}{R}\right) \text{ co } \tfrac{1}{2}n(n+1)d_1. \end{aligned} \right.$$

Therefore

$$\frac{2T}{r_1} = \text{ta}\,\tfrac{1}{2}I \left\{ \begin{array}{l} 2 \\ -\text{co}\,d_1 \\ -\tfrac{1}{3}\text{co}\,3d_1 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ -\dfrac{2}{n(n-1)}\,\text{co}\,\tfrac{1}{2}n\,(n-1)d_1 \\ -\left(\dfrac{1}{n}-\dfrac{R}{r_1}\right)\text{co}\,\tfrac{1}{2}n\,(n+1)d_1\,. \end{array} \right.$$

$$+ \left\{ \begin{array}{l} \text{si}\,d_1 \\ +\tfrac{1}{3}\,\text{si}\,3d_1 \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ +\dfrac{2}{n(n-1)}\,\text{si}\,\tfrac{1}{2}n\,(n-1)\,d_1 \\ +\left(\dfrac{1}{n}-\dfrac{R}{r_1}\right)\text{si}\,\tfrac{1}{2}n\,(n+1)\,d_1 \end{array} \right. \qquad \text{(I).}$$

6. Computation of the Length of the Curve:—

Consider the n equal chords of the initial transition curve. Indicating the angle between any chord and the tangent at its extremity to one of the equal arcs by d', which is the *deflection angle* for that curve for that length of chord, and equal to half the central angle subtended by the arc, we have for the n arcs the deflection angles d'_1, d'_2, . . . d'_n equal respectively to $\tfrac{1}{2}d_1$, $\tfrac{1}{2}d_2$, . . . $\tfrac{1}{2}d_n$.

It follows at once that the angle which the tangent at the end of the m^{th} arc makes with the initial tangent is

$$m\,(m+1)\,d'_1.$$

Therefore the m^{th} chord makes with the initial tangent the angle

$$m^2\,d'_1.$$

Hence the angle of contingence (intersection angle) for the whole of the D° curve is

$$I^\circ - n\,(n+1)\,d_1.$$

The length of the D° portion of the curve in stations is then

$$L = \frac{I^\circ - n\,(n+1)\,d_1.}{D^\circ} \qquad \text{(II).}$$

7. Computation of the deflections from the tangent necessary for locating the transition curves :—

(1.) Consider the initial Transition Curve. It is desired to determine the angle which the long chord from the *P. C.* to the end of the m^{th} arc of the transition makes with the initial tangent.

Project the chords of the curve on the initial tangent and on a line normal to it as axes of x and y respectively.

The angle which the r^{th} chord makes with the initial tangent is (§ 6)

$$r^2 d'_1.$$

Hence the projections of the r^{th} chord are

$$\Delta x_r = c \operatorname{co} (r^2 d'_1) \quad \text{and} \quad \Delta y_r = c \operatorname{si} (r^2 d'_1).$$

The co-ordinates of the end of the m^{th} chord are then

$$x_m = c \sum_1^m{}' \operatorname{co} (m^2 d'_1) \quad \text{and} \quad y_m = c \sum_1^m{}' \operatorname{si} (m^2 d'_1).$$

If then V_m be the above required angle, we have

$$\operatorname{ta} V_m = \frac{y}{x} = \frac{\sum_1^m{}' \operatorname{si} (m^2 d'_1)}{\sum_1^m{}' \operatorname{co} (m^2 d'_1)}. \qquad \text{(III).}$$

From which the required deflections are to be computed.

(2.) Consider the terminal Transition Curve. It is desired to determine the angle between the terminal tangent of the D° curve and the long chord determined by the end of the D° curve and the end of the m^{th} chord of the transition curve.

The angle which the r^{th} chord of the transition curve makes with the terminal tangent to the D° curve is

$$\left\{ (2r-1)n-(r-1)^2 \right\} d'_1.$$

Hence the projections of the r^{th} chord on the terminal tangent of the $D°$ curve and a line normal to it as axes of x and y respectively are

$$\Delta x_r = c \text{ co} \left\{ (2r-1)n - (r-1)^2 \right\} d'_1;$$

$$\Delta y_r = c \text{ co} \left\{ (2r-1)n - (r-1)^2 \right\} d'_1.$$

The co-ordinates of the end of the m^{th} chord referred to these axes are then

$$x_m = c \sum_1^m{}' \text{co} \left[(2m-1)n - (m-1)^2\right] d'_1;$$

$$y_m = c \sum_1^m{}' \text{si} \left[(2m-1)n - (m-1)^2\right] d'_1.$$

We have therefore for the required angle

$$\text{ta } V_m = \frac{\sum_1^m{}' \text{si} \left[(2m-1)n - (m-1)^2\right] d'_1}{\sum_1^m{}' \text{co} \left[(2m-1)n - (m-1)^2\right] d'_1}. \qquad \text{(IV).}$$

From which the deflections for running in the terminal transition are to be computed.

8. Computation of the External:—

From the figure it is seen that the external distance (the disrance from the intersection I to the mid-point of curve) in terms of the external for a simple $D°$ curve is

$$E = E_R + \text{se } \tfrac{1}{2}I \sum_1^n{}' z. \qquad \text{(V).}$$

The effect of the transition on the location is then to move the curve *in* an amount

$$\text{se } \tfrac{1}{2}I \sum_1^n{}' z.$$

9. The five formulae above given complete the solution of the transition curve in all its generality. They are in their

present forms worthless to the engineer. It is now necessary to fix the value of d_1 and of r_1 from practical considerations of the effect of change of curvature upon moving trains, and of the length of arc of uniform curvature necessary for a train moving with given velocity to acquire steady motion. These data had best come from engineers who have studied more closely than the writer the practical aspect of the question. We shall make assumptions here, upon these matters, which while they are open to question, serve admirably the purpose of illustrating clearly the practical applicability of the general formulae.

From what data I have been able to gather it appears that a change of curvature of *one degree* per hundred feet of track is not injurious nor objectionable, that the transition from a tangent to a *one degree* curve is not objectionable and that therefore a 1° curve needs no easement. On this basis the following system of transition curves is designed.

10. It is required to unite two tangents meeting at I° by a D° curve and to pass from the tangents to the curve by arcs of 50 feet whose curvatures are respectively those of 1°, 2°, 3°, . . . n° curves, n being the number of integral degrees in D°. In other words $D^\circ = n + m/60$, where m is the number of minutes in excess of the integral number of degrees in D. I shall not develope the system beyond $D^\circ = 8^\circ$, this being the maximum for first-class roads.

(*1.*) Tangent:—

In (I.) put $r_1 = 5730$ and $d_1 = \frac{1}{2}^\circ$. The formula for the tangent then becomes,

$$T = (R + a)\,\mathrm{ta}\,\tfrac{1}{2}I + 50n\left(1 - \frac{n+1}{2D}\right).$$

We may change the formula for T into that for the tangent to a simple D° curve and apply a correction, thus

$$T = T_R + \left\{a\,\mathrm{ta}\,\tfrac{1}{2}I + 50n\left(1 - \frac{n+1}{2D}\right)\right\}.$$

In which a is to be taken from the following small table giv-

* The Author's Note:—
This table is all wrong.
The value of $a = \Sigma z$ is given by the following formula/

$$a = \Sigma z = \frac{n^2(n+1)^2}{18}\left[\frac{1}{3}\frac{n-1}{n(n+1)} + \frac{1}{n} - \frac{1}{D}\right] \text{ Nearly}$$

$$= \frac{n^3}{50} + \frac{mn^2}{1000} \text{ practically.}$$

ing values of a correctly to tenths, corresponding to the values of D° for each minute up to 8°.

* TABLE I.

D	1°	2°	3°	4°	5°	6°	7°
00′	0.0	1.8	5.6	12,0	50,5	65,3	100,1
10′	0,4	1,7	5,5	11,8	50,1	64,9	99,6
20′	0,4	1,7	5,4	11.6	49,[illegible]	64,5	99,0
30′	0,4	1,6	5,3	11,4	49,5	64,1	98,5
40′	0,4	1,6	5,2	11,2	49,3	63,7	98,0
50′	0,4	1,5	5,1	11,0	49,0	63,4	97,5
Diff	0.0	0.05	0,1	0,2	0,3	0,4	0,5

(2.) Length:—

The whole length of the curve in chains, in terms of the length of the simple D° curve is

$$L = L_R + n - \tfrac{1}{2}\,\frac{n(n+1)}{D},$$

$$= \frac{I - \frac{1}{2}n(n+1)}{D} + n.$$

The excess of length over that of the simple curve is in chains

$$n - \frac{n(n+1)}{2D}.$$

Which is independent of I. The effect of the system of curves *is to shorten the D° curve*, while the increment in length of the whole curve over that of the simple D° curve *is not* increased by the length of the transitions.

(3.) The deflection angles for setting the Transitions;—

(*a.*) The Initial Transition:—

Putting the value $d'_1 = \frac{1}{4}$, in (III.) we deduce the following deflections from the initial tangent required to run in the initial transition, as tabulated in the line marked V_i.

From these results we see that the chords after the first, subtend at $P.\ C.$, angles which are in arithmetical progression with

...ney 1891

Table I

D	1°	2°	3°	4°	5°	6°	7°
0′	0	0.1	0.4	1.1	2.2	4.0	6.2
10′	0	0.1	0.5	1.3	2.5	4.3	6.7
20′	0	0.2	0.7	1.4	2.8	4.7	7.1
30′	0.1	0.2	0.8	1.6	3.1	5.1	7.8
40′	[illegible]	0.3	0.9	1.8	3.4	5.5	8.3
50′	0.1	0.4	0.9	2.0	3.7	5.9	8.8

a common difference of ten minutes. These angles are tabulated in the line marked D_e. They are easy to remember and are therefore turned off with facility.

To recover the tangent at the end of the m^{th} chord of the transition, deflect from the long chord to *P. C.* the angle

$$D_t = \tfrac{1}{4}m(m-1) - V_i.$$

Since $\frac{1}{4}m(m+1)$ is the angle (§ 6) which the tangent at the end of the m^{th} arc makes with the initial tangent. These angles are tabulated in the line marked D_t.

TABLE II.

n	1	2	3	4	5	6	7
V_i	0° 15′	0° 37′½	1° 10′	1° 52′½	2° 45′	3° 47′½	5° 0′
D_e	15′	22′½	32′½	42′½	52′½	62′½	72′½
D_t	15′	52′½	1° 50′	3° 7′½	4° 45′	6° 42′½	9° 0′
C	50.0	100.0	149.9	199.9	249.8	299.5	349.0
x	50.0	100.0	149.95	199.83	249.55	298.92	347.79
y	0.218	1.09	3.05	6.54	11.99	19.80	30.42

(*b*.) The Terminal Transition:—

To locate the terminal transition in the regular way the transit is set at the end of the *D* curve and the ends of the chords of the transition set successively by deflections from the terminal tangent of the D curve. Here each value of n gives a different set of deflections.

The table below shows in a convenient and condensed form the numerical values of V_t for the curves from 1 to $n = 7$ chords.

1	0° 15′						
2	0° 30′	0° 52′½					
3	0° 45′	1° 22′½	1° 50′				
4	1° 0′	1° 52′½	2° 35′	3° 7′½			
5	1° 15′	2° 22′½	3° 20′	4° 7′½	4° 45′		
6	1° 30′	2° 52′½	4° 5′	5° 7′½	6° 0′	6° 42′½	
7	1° 45′	3° 22′½	4° 50′	6° 7′½	7° 15′	8° 12′½	9° 0′

TABLE III.

These angles (V_t) have been computed by formula (IV.) in which we put $d'_1 = \frac{1}{4}$, and gave m values from 1 to n corresponding to values of n from 1 to 7.

Otherwise the total deflection for the terminal transition may be turned off at once from the TABLE II, being the value of D_t, and the *P. T.* set by measuring the long chord of the transition *C*, as computed below and found in the line *C* of TABLE II; transferring to *P. T.* and running the terminal transition in backwards by the angles V_i of TABLE II, which renders the TABLE III superfluous.

11. Computation of the tangent-offsets to the corners of the chord polygon of the transition.

Projecting the polygon on the initial tangent and a normal to it as in § 7, we have for y, the offset to the curve at a distance x from the *P. C.*,

$$y_m = 50 \sum_1^m \text{si} \tfrac{1}{4}m^2; \qquad x_m = 50 \sum_1^m \text{co} \tfrac{1}{4}m^2.$$

And for the long chord

$$C_m = y_m \csc V_i = x_m \,\text{se}\, V_i.$$

These values have been computed and tabulated in TABLE II for the values of m from 1 to $n = 7$.

The intermediate points on the curve are best set from the chord ordinates which are for mid- and quarter-chord respectively $\frac{1}{18}m$ and $\frac{1}{24}m$. It is objectionable that the regular stations cannot be set with the transit nor offset from the tangent, but it is so easy to set points on the transition at intervals of $12\frac{1}{2}$ feet that the regular stations are put in without difficulty.

12. It has been the writer's purpose in the present paper to develope the true transition curve generally, as this has not I believe been done. The numerical system of curves has been computed to illustrate the simplicity of the final results, and not to urge its practical adoption over any other system which may be proposed. Thus it may be better to approach the curve by a succession of 25 foot arcs, beginning at the tangent with a 30′ curve and letting the progression be 30′, 1°,

1° 30′, etc., or to let the chords be 100 feet and the progression as assumed in the system above. Whatever be the length of chord, the initial curvature and the change of curvature, the numerical results can be brought to the same degree of working simplicity as shown above. Even though Mr. Wellington's promised system of transition curves based upon the cubic parabola should prove vastly superior to such as offered here, it is still well that this work should be done, for until it is, and its practicability or impracticability established, engineers will long for the approach to a curve whose curvature increases in arithmetical series for arcs of equal lengths.

13. The Transition Spiral:—

Before leaving the subject I desire to add a few remarks regarding the true geometrical curve to which the Transition Curve defined in the above is an approximation.

Geometrically the Transition Curve is then that curve whose curvature is directly proportional to the length of the curve, or whose radius of curvature at any point is inversely proportional to the length of the arc measured from that point to some fixed point on the curve. Writing this fundamental property in the usual symbols, we have

$$\frac{1}{\rho} = m\,s.$$

Hence

$$d\varphi = \frac{ds}{\rho} = m\,s\,ds;$$

φ being the contingent angle.

Integrating this expression between the limits 0 and s, we have for the intrinsic equation to the curve, in terms of the length of the arc measured from the origin, and the angle which the tangent at any point makes with that at the origin,

$$\varphi = \tfrac{1}{2} m\,s^2.$$

Refer the curve to rectangular co-ordinates through the origin, the tangent there being the x-axis and a normal to it the y-axis.

Then

$$dx = \text{co}\ \varphi\ ds,$$
$$= \frac{\text{co}\ \varphi\ d\varphi}{\sqrt{2m\varphi}},$$
$$= \text{co}\ (\tfrac{1}{2} m\ s^2)\ ds.$$

Also

$$dy = \text{si}\ \varphi\ ds,$$
$$= \frac{\text{si}\ \varphi\ d\varphi}{\sqrt{2m\varphi}},$$
$$= \text{si}\ (\tfrac{1}{2} m\ s^2)\ ds.$$

Thus the co-ordinates of a point on the curve in terms of s are

$$x = \int_0^s \text{co}\ (\tfrac{1}{2} m\ s^2) ds; \qquad y = \int_0^s \text{si}\ (\tfrac{1}{2} m\ s^2)\ ds.$$

These integrals we are not able to evaluate in finite terms. It is easy to see from the definition of the curve that its ultimate destination is an asymptotic point, and that its Cartesian equation is of infinite degree.

To determine the co-ordinates of the asymptotic point we must make the superior limits in the above integrals infinite; then, treatises on the Calculus show that

$$\int_0^\infty \text{co}\ (\tfrac{1}{2} m\ s^2) ds = \int_0^\infty \text{si}\ (\tfrac{1}{2} m\ s^2) ds = \frac{1}{2}\sqrt{\frac{\pi}{m}}.$$

The co-ordinates of the ultimate points are therefore equal. The curve itself is a very beautiful spiral resembling the sign used for indicating integration, the origin is a point of inflexion beyond which the curve repeats itself in the opposite direction, terminating in an asymptotic point on the same straight line through the origin and the other terminus.

14. Mr. Wellington bases his system of Transition Curves upon the geometrical curve which enjoys the property of having its curvature at any point directly proportional to the distance of the point, measured along some fixed tangent, from the point of contact. Referring the curve, as above, to the tangent as x-axis and the point of contact as origin; then the curvature at any point is directly proportional to its abscissa x,

or the radius of curvature is inversely proportional to x.

This curve is well known in geometrical text-books to be the *elastica.* The above defining property was first shown by James Bernoulli. This property written in symbols is

$$\rho x = m.$$

Putting in the regulation value for ρ in terms of x and y, we have

$$x = \frac{m\dfrac{d^2x}{dy^2}}{\sqrt{\left(1+\dfrac{dx^2}{dy^2}\right)^3}}.$$

Multiplying by 2 dx and integrating, we have

$$x^2 = C - \frac{2m}{\sqrt{\left(1+\dfrac{dx^2}{dy^2}\right)}}.$$

To evaluate the constant C we notice that when $x=0$, then $\dfrac{dx}{dy} = \infty$, and therefore $C=0$.

Finally

$$y = \int \frac{x^2\,dx}{\sqrt{(4m^2 - x^4)}},$$

which is the equation to the curve in terms of an elliptic integral, and which again cannot be evaluated in finite terms.

Rolla, Mo., March, 1890.

THE BEGINNINGS OF MATHEMATICS.

By Prof. W. B. Richards.

Qui currit, legat.

I.

The human mind is not content with the fact; it desires to know the process. The youth who vivisected the bellows in order to discover the cause of its action is a type of his kind. "Nothing is covered that shall not be revealed;" this is not the least of the joys that await the faithful. To unravel the tangled skein of mysteries that weave us about, to bring the hidden to light, to illumine the dark places, to rescue from the unknown some of its treasures—this has been the incentive that has animated man in every age, has raised his "clear spirit" to "scorn delights and live laborious days," has urged him forward from point to point of achievement. It is the spirit that inspired the wonder-working mind of Aristotle, lit "the lonely lamp of Erasmus" and smoothed out "the restless bed of Pascal." The thirst of discovery, like Io's gad-fly, will not let man be; it goads him like Jove's ill-fated favorite into restless wanderings through all the obscurest corners of the earth, and all the trackless fields of intellectual research. It wafted the ships of Columbus toward the western world, led De Long to his frozen grave in the wastes of Siberia, and has lately sent Stanley across deserts, over mountains, through savage tribes to the heart of the Dark Continent. Nor has its influence been less present in the intellectual world, than in the sensible. Needless to call the honor roll of great minds that attest it. The mind knows no rest. The horizon of its aspirations recedes as it is approached. Its stopping points are only night-camps, wherein it prepares for the morrow's march. It may need to intrench itself against the powers of doubt and unreason, but it does not find an abiding place. It never reaches the end. Nor can it. Truth is infinite. A Newton about to die ptotests sorrowfully that he "has only been picking up pebbles beside the great sea-shore." At the same time that we aspire to add to the world's mental enrichment, it cannot fail, it seems to me, to be both helpful and interesting to consider the steps by which what we have has been won. It is for this reason that we design to set down in a shape suited to general readers some account of the beginnings of that science which ccntains within itself the germs of all other sciences.

The student of the Mathematics of to-day may well be astounded at the vastness of the field which is open to him, at the multitude of directions in which investigation has been pushed, and the wonderful achievements that have been made in each. If, in the midst of his gratulation upon modern attainments, there is, however, danger of his conceiving a contempt for the lesser success of earlier workers, he should reflect that, if we see farther into the mysteries whose solution has been the problem of all ages, it is not necessarily because our intellectual vision is so much more acute, but partly, at least, because, as has been said, "we stand upon the shoulders of giants." The superiority of modern Mathematics over the ancient does not so much arise from a comparison of the body of truth acquired, as it follows from the discovery of new methods—the improvement in *technique*, as it were. We do not build structures larger than the Pyramids, but we know how to build them more easily. One who reads the history of Mathematics wonders not more at the advancement which the moderns, having all the experience and the result of the labors of their predecessors to guide them, have made, than at the great fund of mathematical knowledge which the old Greeks were able to master with the means at their disposal. It was a pure triumph of unassisted mind. Imagine yourself deprived of all knowledge, if not quite of Algebraic processes, yet of Algebraic notation, which is a chief element of the strength of Algebra; conceive yourself unable to use a symbol for a quantity or a complex combination of quantities, to use + or − or to write an equation; think how greatly the difficulty of an abstruse problem would be increased. Yet with such negative disadvantages did the ancients work. They were too busy, getting out the rich ore from the mine that had been opened to them to stop to sharpen their tools or to exchange them for new ones. Where they advanced laboriously in their rude but forceful way, we touch off a little Calculus under the obstacle and—piff!—it is gone. But what treasures did they uncover! What might they not have done if their *finesse* had been equal to their strength! To one who has not previously considered the subject, the antiquity of most of the Mathematics ordinarily taught in our colleges is surprising. The Elementary Geometry is practically as left by Euclid twenty-two hundred years ago. In England translations of Euclid's work are used, while on the Continent, and in this country, the text books are adaptations of his work. Algebra is a comparatively modern growth, having been introduced into Europe in the thirteenth century, while its symbols were all invented in the last four hundred hears. The solution of equations of the second degree, with general co-efficients. however, was effected by the Hindoos certainly as early as Aryabhata in the 5th century, A. C., and perhaps earlier. Our Analytic Geometry is the product of the wedding of the Geometry of the Greeks and the Algebra of the Hindoos, brought about by Descartes in the first half of 17th century, but the method of analysis may be traced back to the school of philosophers immediately following Plato, while most of the properties of the conic sections were known to Apollonius—the "Sublime Geometer," as he is called by Geminus—and are announced by him in his "Treatise on Conics," (3d century B. C.) The Infinitesimal Calculus could not arise without Algebra and its invention was the second great fruit borne by that science in the seventeeth century; but the germs of its fundamental analysis are to be found in Archimedes' "Method of Exhaustions" and many of the practical problems to which it is applied—such as the quadrature of surfaces, the cubature of volumes, the calculation of the value of π were successfully attacked by the early Greek mathematicians. One of the greatest of modern mathematicians pays a just tribute to one of the greatest of the ancient, when Leibnitz says, "Those who can understand Archimedes admire less the discoveries of the greatest moderns." Even what is known as Modern Geometry is not altogether so recent as might be imagined from the name. Some of the fundamental theorems concerning transversals are enunciated and demonstrated by Pappus who lived in the 4th century A. C. This is six huudred years later than Euclid's Geometry, but only in the most relative sense could it be called Modern. The same writer enounces without demonstrating the theorems connecting the surface and volume generated by ths revolution of a plane curve about an external axis in its plane with the path described by the centroid of the perimeter and area respectively—usually cited as Guldin's Theorems. It is not our purpose to institute any invidious comparison of the merits of the ancient and of the modern mathematicians, similar to that which in the field of letters fomented the celebrated controversy that two hundred years ago divided English men of learning into hostile camps, but a suggestion of the respectable and even admirable attainments of antiquity may stimulate an interest in the discussion which we propose.

Mathematics is a comprehensive term which imports very different things to different people. To the child just beginning to wrestle with arithmetic, it probably means the multiplication table and an outlying unexplored territory of unknown dimensions. To the average "Young Ladies' Seminary" 'young lady' it means,—or it used to mean, for late years have shown an improvement in this respect—Arithmetic, some dalliance with Algebra, the memorizing of certain portions of Euclid, and perhaps a faint suspicion of Trigonometry—the prevailing idea of this subject being that it is something in the back of Geometry. To each of us, perhaps, it means as much as he knows, a good deal that he suspects, and a great deal more that he would like to know. Mathematics is the generic name popularly applied not merely to the labored and difficult processes of a Newton or a Laplace but as well to the first slate-scribblings of the primary scholar. Including, thus, that which pertains so closely to our earliest mental feats, we might suppose that in order to get to the beginnings of Mathematics it would be necessary to go back very nearly to the beginnings of things—to the time "when Adam delved and Eve span." Our introduction to numerical calculation occurs at so early a stage in our experience, it is so nearly contemporaneous with the utmost backward reach of memory, that it is not strange if our proneness to judge others by ourselves leads us to infer that the same notions came to primal man at a correspondingly early period in his history—indeed were a natural and necessary outgrowth of his mentality.

That these presumptions are erroneous is sufficiently indicated by the facts which we are about to adduce. Percepts antedate concepts, The mind of early man doubtless proceeded, like that of children, by the inductive method—ascending from the cognition of particular facts to the intuition of general laws. We think by means of pictures more or less clearly photographed on the mental curtain. These pictures are either of the things themselves, or, especially in the case of an educated person, of the names or the symbols of the things thought. Try to recall some familiar quotation, and memory, repeating the original process of thought, will bring before the mental vision either the scene or action described, or, it may be, the lines as printed in the text from which you learned it. Say over those lines in which Virgil tells how Priam fell at the foot of the altar that streamed with the blood of his slaughtered son, and either you shall see the sad scene enacted before your mind's eye or, it may be, there will pass before you the lines of some old dog-eared copy of the Augustan epic, from which, in school-boy days, you droned out your task. It is easier to think of concrete, sensible objects, than of the abstract, because of the former we may make a definite picture. We take advantage of all this in educating children. We fill their books with pictures. In teaching a cnild the rudiments of Arithmetic we ask him at first not "How much are 2 and 3?" but "How many apples are 2 apples and 3 apples?" or "How many marbles are 2 marbles and 3 marbles?" enabling him to make a picture out of the problem, and asking him to tell just what he sees, Thus we lead him inductively to the notion that two and three make five independently of the nature of the substance numbered. The unaided human mind, working out its own destiny, it may be assumed, made its tedious progress over a similar track. The primal man, as he drove in succession two pairs of oxen into a corral, was aware of a quadrupleness of objects, though he did not as yet separate in his mind the number from the things numbered. It was a long step from this single experience or a great number of such experiences to the dawning of the abstract law that two and two make four—whether it be of oxen or what not. When the idea of abstract number had begun to stir, the next thing would be to find names for the numbers, and no great advance in such thought could be made until the invention of words to indicate number made mental combination of numbers possible. That we are correct in inferring that this development of the idea of number was not not neccessary nor immediate, is shown by the fact that tribes are to be found at the present time which have not attained such advancement. The Chiquito, the language of the natives of Eastern Bolivia, is said to be absolutely destitute of numerals. Counting is unknown to them. The word that comes nearest to meaning "one" is that which signifies "itself" or "the same;" beyond this point the mathematical ability of these children of nature does not go. Their mind surrenders at the difficulty of grasping so large a number as two, and expresses it and all greater multitude by the indefinite word for "many." The Papuans of Torres Strait have names for only one and two. The Bushmen of Australia are scarcely more advanced. Their numeral system ends at three. The traveler, Pelleschi, relates that on the plains known as El Gran Chaco, in South America, he encountered a chief who could not count his own fingers. Theon, of Smyrna, one of the earliest writers on Arithmetic, states that "Agamemnon was so ignorant of the names of numbers as not to know that he had two feet." The same writer reproaches Pythagoras, Archytas and Philolaus for not having distinguished between "unity" and the number "one"—between the numbers of objects and the objects themselves. "Six oxen," says he, "constitute a sensible number; six is an intellectual number." We thus have abundant evidence that the idea of abstract number was slow in taking shape, and that any adequate system of numerical nomenclature was the result or the concomitant of a considerable mental progress.

Before this had been achieved, when the question "how many?" was asked, the answer would naturally be given by indicating a corresponding number of some other convenient objects. The ready means of replying to such questions seemed made to hand—we use the expression in good faith with no intention of punning—in the ten fingers. Nothing could be more natural than that the untutored savage, in the absence of vocables suited to the purpose, should call the fingers to his aid in conveying numerical ideas. We, to-day, very commonly use the same artifice when we wish to present such information silently, while for the deaf, as is well known, a digital alphabet has been invented. The Arithmetic neophyte is with great difficulty to be restrained from the pernicious habit, when called upon to "do sums" in addition, of using his fingers as a kind of restricted abacus. The Wallachion peasant is said to perform all multiplications above 4×4 with the assistance of his fingers.

The use of the fingers in this connection affords the reason that the numerical systems of all civilized nations are decimal. Traces of the connection between the assumption of ten as a radix and its occurrence as a natural number are to be found in various languages. In the Polynesian, *lima*, i. e. "hand," means five; in the Zulu *tatikitupa*, "taking the thumb," signifies six; in Greenlandish *arfersanek pingasut*, i. e. "taking the other foot three" (the two hands = 10, one foot = 5, and 3) means eighteen. In the Maya dialects of Central America the word for twenty is *hun vinak*, one man; that is the number of fingers and toes belonging to one person. Similarly in New Caledonia the word for man means twenty, while "five men" means one hundred. In English, likewise, the old fashion of counting by scores smacks of the same origin—with which we may compare the French way of expressing ninety-three, for instance, by *quatre-vingt-treize*, four twenties plus thirteen. The word that we use for the figures of a number—digit—is directly from the Latin *digitus*, a finger, and indicates the same connection.

These primitive movements in the direction of mathematical cognition are only to be considered the beginnings of Mathematics in the same relative sense in which the first stone thrown was the beginning of ballistics, or the first tree hewn across a ravine was the beginning of engineering. For the origin of Mathematics as a science we must look to the Greeks—to that prolific national mind to which all the learning of the West may be traced as to its spring. In the domain of learning all roads lead back to Greece. The beginnings of whatever is worthiest in Literature, in Philosophy, in Art, in Science were made by the marvelous people who have given us the epic of Homer, the Logic of Aristotle, the sculptures of Phidias and the Geometry of Euclid. No tribute of admiration can be too high to express a just sense of our indebtedness for the imperishable legacy which we have inherited from them. There is no part of the world's wealth to-day with which it might not better part than with its attainments in those departments of mind in which the earliest impulse, and frequently the most lasting monuments, were the products of Hellenic thought. The distinctive features of the Greek intellect were just those which were best suited to grapple with the problem that confronted them. This problem was two-fold—the extension of knowledge, and the formation of truth into a connected system. The same problem it may be said, confronts all periods. True. But the circumstances in which the Greeks approached it were not the same as those in which later times, enjoying the fruits of their labors, have succeeded to it. In the first direction only a beginning had been made, while the second was yet unattempted. To each branch of this task the Greeks brought an especial fitness. The most prominent characteristics of their mind were the instinct of investigation and a genius for form. The first finds its conspicuous development in Socrates, who declared, "Φιλοσοφοῦντά με δεῖν ζῆν ἐξετάζοντα ἐμαυτόν τε καὶ τοὺς ἄλλους,"* while both attain their consummate flower in Aristotle. The enthusiasm which came with the birth of philosophy stimulated inquiry in every direction in which truth was likely to be its reward. In the confident words which Bacon used of himself, they "took all knowledge to be their province." They were not content with any one-sided development of a single branch of learning. Older peoples had attained some advancement in special fields of knowledge. The Assyrians, as we shall see, had reaped some results from centuries' study of the stars, and the Egyptians possessed the rudiments of Geometry. But the comprehensive intellect of the Greeks proposed to itself as its goal nothing less than the sum of all knowledge.

The genius for form is the germ of that sense of beauty, both real and ideal, which showed itself in every phase of their life, which was the informing spirit alike of their art and of their ethics. This element of order in conjunction with the iuspiration of inquiry produced for the first time a philosophic spirit. The results of their investigation were to be compared, digested, systematized. More; facts were no longer the end of their search; they went further and sought principles. It no longer sufficed to ask "*an sit*;" they must know "*cur sit.*" They did not scorn to learn what they might from others; but they seemed to have the power, like the fabled Midas of their own legend, of transmuting all they touched to gold. The scientific method, which appears for the first time with them, was the agent of this alchemy.

While the history of the science of Mathematics finds its appropriate point of departure with the Greeks, a study of earlier culture is necessary to an understanding of the state of knowledge at the time their intellectual activity began, and the share in its subsequent development contributed by other nations. Either prior to Greek civilization or contemporary with it and running paralel to it, we may note three early seats of culture

*I must needs spend my days philosophizing, examining both myself and others.

from which the outcome of Greek thought was influenced: viz, Babylonia, Egypt, Phœnicia. We do not include India in the list, because, while the Hindoos justly claimed a venerable antiquity for their civilization, there was no communion between them and the Greeks until after the conquest of Alexander (325 B. C.), and they made no impress on the Mathematics of the West until Algebra began to be studied in the Middle Ages. Each of these countries contained a considerable population and the first two were the homes of powerful empires.

Movements of population do not occur by chance or at the dictation of caprice; they are determined by causes usually not difficult to discover. Since the unfortunate mis-step—it would be disrespectful to use a harsher word—of our first parents, the chief energies of man have been directed toward an attempt to escape the curse of labor then pronounced upon Adam and his seed. He is ever seeking to live, either by the sweat of somebody else's brow (which is called "genius"), or by as little as possible of his own (which is popularly known as "talent"). The agitation now making by Labor organizations looking toward a reduction of the hours of labor, is merely a fresh manifestation of a well-nigh world-old spirit. People desire, they have always desired, to get a maximum of existence out of a minimum of exertion. Hence in the early days

"When the world was all before them, where to choose"

tribal communities would seek for habitations.lands in which the climate was least rigorous and changeable, where Nature had provided most generously for their herds, and where the soil responded most kindly to tillage. Observe how population tended to settle down into southern peninsulas, as if it were a molten mass operated upon by gravity. The force which was actually at work acted just as surely. It was the attraction of a clearer sky and a more genial sun. The fact too that migrating parties found themselves in a kind of *cul-de-sac* with the sea hemming them on all sides but one contributed toward stopping their wanderings. Let us loiter from our subject long enough to say—what may have been stated before—that it may be roughly laid down as a law, not, however, to be too strictly interpreted, that the civilization of a primitive people varies directly as the ratio of their sea-coast to the total area of their country. We may cite as examples on the one hand Greece, Italy; on the other, Africa. The reason is not far to seek. Navigation in early times was far in advance of any system of land travel. The sea was a means of communication, connecting, rather than dividing, distant peoples; while those who dwelt far inland were cut off from association with their fellows and failed to get that sharpening of ideas which comes from mental attrition.

It is in accordance with the natural law to which we have referred that the rising of the traditional "curtain of History" discloses the two oldest civilizations flourishing in the rich valleys of the Nile and of the Tigris and Euphrates. The latter district, close to what legend proclaims the cradle of the human family, was at an early date inhabited by a Turanian tribe, akin to the Magyars of Hungary, the Lapps and Finns of the Arctic circle, and the Tartars of the Russian steppes. At first a nomadic people they became later builders of cities, and excavations in recent years have brought to light interesting specimens of their architecture. Indeed their architectural propensity is represented as having proved a source of the direst misfortune both to them and the world at large; for on "the plain of Shinar" they essayed to build a heaven-reaching tower—an act of presumption which brought upon them—and us—the confusion of tongues. The name of the place of this unfortunate experiment was called Babel (confusion), from which, according to a popular etymology, the name Babylon is derived. The southern branch of this people, the Accads, came in contact with a Semitic tribe who in time became the dominant portion of the mixed population. These were the Chaldeans, from whose chief city, Ur, the biblical records represent Abraham as emigrating to the land of Canaan. To the north, in the higher lands dividing the waters of the great rivers, dwelt a kindred tribe, the Assyrians, and the history and art and science of the two peoples are closely interwoven. We have spoken of the whole territory as Babylonia for the sake of a single name, but their common learning is more usually styled Assyrian.

The nature of the mental product of these early workers is what might be expected from their habits and environment. Chiefly a pastoral people, they had their wealth in flocks and herds. So we find in Genesis contention arising between the herdsmen of Lot and of Abraham "because the land was not able to bear their flocks." The climate and their occupation made them dwellers in the open air. They learned to guide themselves by means of the stars across the vast level or billowy tracts of land, lying before them like a sea. There were no printed volumes to read, but the newly edited book of Nature in all its freshness, invited and compelled their study It is not strange that the herdsman, lying on his back, while the cattle grazed, should have attempted to decipher the mysteries of that brilliant page unrolled each night before his wondering vision; that he should learn to look for the coming of the stars as of some distant, supernatural companions, and that from a repeated contemplation of the heavenly bodies he should grow to reverence and adore them as divinities. Thus natural curiosity, material interest, and religious veneration, all conspired to make the Assyro-Babylonians students of the stars, and brought it about that their chief attainments in knowledge were in connection with Astronomy. The inception of the study of Astronomy occurred among the Accads, to whom their observatories were instruments at the same time of science and of religion. Their successors followed the impetus thus given. The stars were numbered and named, and a chart of the heavens was constructed. A calendar was formed in which the year was divided into twelve months of thirty days each. To supply the deficit from the actual number of days in a year, a month was intercalated every six years, and the priests were charged with the insertion of other months at such periods as were necessary. Eclipses were observed, and a record of them kept. They are said to have invented the sun-dial and clepsydra; also the lever and the pulley. The needs of the extended commerce which they gained in later years gave rise to the invention of weights and measures the origin of which are sometimes attributed to the Phœnicians.

The founders of the ancient civilization in the valley of the Nile, it should scarcely be necessary to say, did not belong to the African race. Their own traditions assert them to have been the original inhabitants of their land, but the evidence that they were of Asiatic stock is conclusive. Their language is what is known as a member of the Hamitic family and bears such an analogy to the Semitic and Aryan tongues as to indidate a relationship, if not a common origin. The Egyptians gloried in the antiquity of their institutions. The darkness from which they emerge into history sheds no ray of definite light upon the steps of their advancement, but the gigantic structures of their erection, standing amid the encroaching sands of the Sahara, are mute but indisputable witnesses of their craft. The gloomy imagination of this venerable people seemed to take a morbid pleasure in its own awe. The inferiority of man to the powers of nature, always borne in so strongly upon dwellers in a tropical region, weighed upon their spirits until whatever by comparison showed the littleness of man, like a basilisk, attracted while it terrified them. The builder's instinct was present in them as in the Assyrians, but the object which they set before themselves was not—as with the Greeks—to please with the beautiful, but to impress with the colossal, the huge, the awe-inspiring. Magnitude of dimension, not grace of outline, was the salient feature of their architecture. The Pyramids and the Parthenon tell the whole story of the minds that conceived them. Like the Babylonians, they sought to "unwind the process of the stars," and produced a calendar similar to that of their Semitic kinsfolk. What is most to our purpose, they laid the first few rude stones from which the Greeks constructed Geometry.

Sciences are not evolved from the human consciousness by definite design. One does not shut himself up in his study, and say "I will straightway develop me a science of chemistry, of engineering, of government, or of what not." No; they arise in response to practical necessity, and grow with the extension of experience and of thought in the direction suggested. First the fact, the suggestion, the experiment, it may be; then the theory; when the inductive process has gone so far, deductive demonstration begins, and the united body of truth becomes a science.

Geometry, the first branch of Mathematics to be developed, bears in its name the stamp of its practical origin; as it came to the Greeks it was simply "earth-measuring." The Assyrians had no Geometry because they had no need for it. Occasion did not suggest it. They lived a shifting life and had practically unlimited territory at their disposal. They were not dependent for subsistence upon any restricted tract of land, and minute questions of boundary and area did not arise. Why measure the earth when each might have as much of it as he chose? But the Egyptians were a vast populace having fixed seats in a narrowly limited country. They maintained themselves by the cultivation of the prolific fields bordering on the Nile; there was but a relatively small quantity of tillable land to be divided among a great number of inhabitants, and considerations of boundary and measurement assumed a vital importance. We translate from Herodotus, who traveled in Egypt about the middle of the 5th century, B. C., his account of the origin of Geometry. The King referred to was Rameses II, or Sesostris, as he was known to the Greeks, who reigned about a thousand years before the period of Herodotus' travels.

"They (the priests) also said that this king distributed the land among all the Egyptians, giving to each an equal quadrangular portion, and that from this he collected his revenues, requiring the holder to pay yearly rent. If the river, however, cut off a part of any tenant's allotment, he would come to the King and attest the occurrence. The latter would send commissioners to investigate the matter and to measure how much the tract had been decreased, in order that he might pay on the remainder an equitable portion of the prescribed rent. In this way, it seems to me, Geometry was invented and passed over to Greece. The sun-dial, though, and the gnomon, and the twelve parts of the day the Greeks learned from the Babylonians."* Here, then among the early Egyptians we find a practical problem giving rise to the first seeds of Geometry. These seeds bore no fruit on their native soil because the Egyptian cast of mind lacked the qualities necessary to produce a science. They never advanced beyond the meagre rudimentary knowledge, which they possessed as a result of experience and observation, not as a system of demonstrated truth. What they attained, though, is forever notable as constituting the suggestion and and incentive to the Geometry of the Greek.

The Phœnicians, occupying a narrow strip of sea coast along the most eastern border of the Mediterranean, were a Semitic tribe, related in language and race to the Hebrews and the Assyrians. They were a manufacturing and commercial people, bold, alert, enterprising, in short, the Yankees of antiquity. They made glass ware from the sands of the Belus, and extracted from the *murex*, a shell-fish found along their coast, a purple dye which they used in coloring the textile stuffs for the manufacture of which they were famous. The exchange of goods brought them into association with the Babylonians, with whom they had an extensive trade by means of caravan, and with the Egyptians. The Phœnicians were the earliest navigators; their vessels bore the product of their looms all along the shores of the Mediterranean, and even beyond them, past the Pillars of Hercules into the Atlantic, upon which they skirted the western coast of Africa as far south as the Canary Islands, and sailed northward to Cornwall. They exchanged their manufactured articles for the raw products of the peoples with whom they traded. They founded colonies along the northern coast of Africa—chief among these, Carthage—in Sicily, in Spain and elsewhere. It was on a Phœnician ship, sailing to the colony of Tarshish in southern Spain, that Jonah took memorable passage. They came into intimate commer-

*Herodotus, Book II, c 109.

cial relations with the Greeks—themselves skilled and adventurous mariners—and profoundly influenced the early Greek culture.

The ancients regarded the Phœnicians as great inventors; the arts of manufacture, Arithmetic, the invention of weights and measures, and of an alphabet were all attributed to them. More careful investigation has cast a doubt upon their claim to originality. The discoveries ascribed to them seem really to have been borrowed from the Egyptians and the Babylonians. The important work which the Phœnicians did in the advancement of civilization was one of distribution. They were the channel through which the influence of the older civilizations was borne to Greece. They stood, in this way, to the Egyptians and the Babylonians in the same relation which, in later times, the Romans held to the Greeks. The former originated; the latter disseminated. Richer than all the precious stuffs of Tyre and Sidon, they bore to the barbaric West the inspiration of a culture, destined in fitter hands far to outstrip the achievements of its original. So far the progress in the extension of knowledge was the work of Hamitic and Semitic branches of the Caucasian race; their advance was slow and their labors unfruitful because their learning was a lifeless empiricisin. The torch of learning which they bore with but faintly increasing brilliance for centuries, and which lighted only the narrow circle of their personal experience, was soon to be extinguished; but before it expired there was kindled at its flame another, whose transcendent brightness was to illumine all the later course of Time. Ethnic and political forces brought the overthrow of the dominion of these once powerful peoples, and the wave of barbarism which submerged them, buried at the same time their civilization. Their part was done; and new hands were to build upon materials first gotten from them a structure of which they had not dreamed. It was the finer, keener intelligence of the Aryan Greeks acting upon the meagre learning of the older Eastern civilizations with which they gained their earliest acquaintance tnrough the Phoenicians, that gave the world for the first time a science. In a subsequent paper we shall speak of the rise of Greek Geometry.

In preparing this paper the writer has kept constantly at hand the 'Encyclopædia Brittanica," and has availed himself freely of its store of information. He has referred especially to the articles Anthropology, Arithmetic, Astronomy, Babylonia, Numerals, Phoenicia. The Iconographic Encyclopædia--Vol. I Anthropology--and Marie's "Histoire des Sciences Mathematiques et Physiqnes" have also been consulted with advantage.

TALLOW CLAYS.

By Prof. W. H. Seamon.

The "Tallow Clays" found in Missouri are soft unctuous masses of white, grey, pink, yellow, red and occasionally black colors. As taken from the ground they contain a large excess of water which they lose more or less rapidly on exposure to the atmosphere, shrinking and falling to pieces. During this operation which is termed "slacking" by the miners, they lose about 30 per cent. of water, and change color, usually darkening.

The "tallow clays" occur associated with calamine alone, though sometimes small lumps are found near deposits of Blende. We have lately received from Mr. Jno. Kingston, of Granby, Mo., an interesting specimen of an olive colored clay from Sucker Flat, near Joplin, which shows some Cadmium. The miners of Blende apply the term "tallow clay" to some impure Kaolins found near the surface in their localities, but these do not possess those distinctive physical characters so well known to all who have ever handled the true "Tallow Clays."

The "Tallow Clays" are found in layers from a few inches in thickness up to several feet (Geol. Survey of Missouri, '73–'74, p. 419, t. 439,) in lumps of from a few pounds in weight up to several hundred with calamine embedded in them, and in thin streaks in cavities in crystallized calamine. Many miners have told me that they usually find less Calamine in those shafts in which they find large bodies of "Tallow Clay," which observation is confirmed by the experience of the Superintendent of the Granby mining company Sometimes Calamine is found in slabs with and without a banded structure as if it were a pseudomorph after Tallow Clay. The following specimens have been analyzed with the results given below:

No. 1. A specimen from Granby, Mo., given me by Mr. John Kingston, slightly banded with layers of gray and buff tints.

No. 2. A grayish white layer from the mines of the Louisville Mining Company, at Aurora, Mo.

No. 3. A buff colored layer from same piece as No. 2.

	No. 1.	No. 2.	No. 3.
Zinc oxide	64.53	58.27	63.05
Iron oxide	0.07	0.05	1.97
Alumina	0.92	2.15	1.13
Silica	27.12	31.42	25.88
Water	7.36	8.11	7.98
	100.00	100.00	100.00

The undried specimens of Tallow Clay give off water in the closed tube; fuse on charcoal at about 3, always lightening up in color becoming white or ash gray; give the zinc coating when heated in the reducing flame with soda; and are completely decomposed with gelatinization when heated with moderately concentrated hydrochloric acid.

The following analyses represent their average composition. The white varieties which have given such high results are found only in thin streaks and in small amount in the darker colored varieties.

ANALYSES OF

(Specimens

No.	Locality.	Color as taken from the ground.	Color after drying in the air.	S. G.	H_2O at 100°C.	Loss at low red heat, H_2O mainly.
1.	Aurora	White	White	2.91	4.03	3.92
2.	"	"	"	2.92	4.14	4.00
3.	Near Peirce City	"	"	2.95	3.63	3.52
4.	Granby	Gray	"	2.89	4.37	4.13
5.	Aurora	Flesh colored	Light drab	2.77	6.33	8.93
6.	"	" "	" "	2.78	6.53	8.73
7.	Near Peirce City	Cream	Yellowish	---	18.06.	
8.	Aurora	Light brown	Ash gray	2·47	9.38	9.22
9.	"	Yellowish brown	" "	---	10.50	8.40
10.	"	" "	" "	2.57	9.62	8.36
11.	"	Brown	Chocolate	2.99	7.00	10.38
12.	Granby	"	Reddish brown	---	12.50	8.02
13.	Near Peirce City	"	Pinkish yellow	---	10.44	8.19
14.	Aurora	"	Chocolate	2.41	9.50	10.02
15.	"		Reddish brown	2.72	6.76	9.70
16.	"		Brown	2.69.	10.49	8.93
17.	Near Peirce City	Reddish brown	Yellow	2.25	10.78	9.93
18.	Granby	Red	Reddish brown	....	21.58	
19.	Near Peirce City	"	Pinkish		20.15	
20.	Granby		Pale Yellow		16.83	
21.	"		Light brown		12·66	
22.	"		Brown		15.40	
23.	"		White		17.98	
24.	"		Brown		16.74	
25.	"		Pale red		16.11	
26.	"		" "		16.11	
27.	"		Dark brown		14.71	
28.	Aurora		Brown		17.80	
29.	"		Pink		18.30	
30.	"		"		14.25	
31.	"		Brown		16.73	
32.	"		White		12.72	

TALLOW CLAYS.

thoroughly air-dried.)

ZnO	SiO_2	Al_2O_3	Fe_2O_3	CaO	Na_2O+K_2O		Totals.
54.06	35.29	1.64	none	1.80	none	....	100.74
54.92	35.31	1.71	"	0.12	"	....	100.20
56 12	34.82	1.52	"	0.32	undet.	.. ..	99.92
50.35	36.82	1.85	0.01	1.93	traces	....	99.46
35.63	38.26	6.17	4.67	tr.	undet.	No P_2O_5	99.99
36.16	36.90	6.29	4.22	1.02	"	"	99.84
35.64	33.36	11.03	0.80	und.	"	tracce P_2 O_5	99.89
36.38	36.59	4.92	1.89	1.77	none	CO_2 trace NoP²O_5	100.14
42.93	33.86	2.14	0.78	1.07	"	NoCO_2	99.77
30.03	37.34	10.62	2.06	1.36	"	"	99.40
28.56	43.49	5.16	4.38	1.21	"	"	100.17
36.98	31.94	3.05	4.46	2.31	0.810	"	100.07
34.33	34.21	7.91	4.89	0.02	nonc	"	99.99
31.72	39.45	6.44	2.08	1.48	"	"	100.69
32.35	37.11	3.44	9.54	1.06	traces	"	99.95
32.72	36.11	6.26	4.21	1.61	"	P_2O_5.020	100.34
34.40	37.66	3.88	3.36	0.01	none	No P_2O_5	100.02
34.78	30.27	8.78	3.98	0.08	traces	traces of CO_2&P_2O_5	99.47
25.96	34.94	9.92	8.53	tr.	"	P_2O_5.23	99.78
34.83	35.07	14.26					100.99
39.53	37.60	9.40					99.19
38.23	38.43	8.67					100.73
37.12	34.37	10.43					99.90
32.50	40.36	10.42					100.02
32.34	42.08	9.64					100.17
29.94	44.07	9.64					99.79
41.47	36.00	7.11					99.26
36.12	35.20	9.98					99.10
31.54	37.37	12.89					100.10
37.39	34.67	14.37					100.68
37.84	34.45	10.84					99.86
48.93	35.94	2.40					99.99

The above complete analyses have been supplemented by determinations of zinc made by students in this Laboratory of other specimens and the results all tend to show that the "Tallow Clays" are uniformly quite high in Zinc oxide, the air-dried specimens having approximately the following average composition:

Oxide of Zinc	34.57
Silica	58.90
Iron and Aluminium oxides	9.41
Water and other matters	17.12
	100.00

It is of interest, perhaps, to note that a similar clay has been found associated with Zinc ore from Southwest Virginia (see No. 1144 London Chemical News), and in Spain, (see Dana's System of Min. p. 408). I have also been informed that similar clays have been found with Calamine in Colorado but have not been able to verify the statement, From an article on the Zinc deposits of Lehigh, Pennsylvania, published in the Transactions of the American Institute of Mining Engineers, (p. 68, Vol. I), I take the following:

"A compact clay containing from 26.32 per cent. of Zinc, unctuous, and with an eminently conchoidal fracture is believed by Prof. Ripper to be a true mineral."

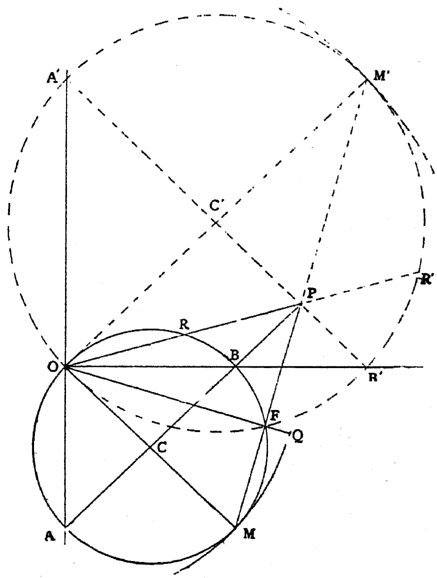
These facts lead me to believe that the "Tallow Clays" are, or will be found with every deposit of Calamine throughout the world.

ON THE CONTINUOUS CONSTRUCTION OF THE ELLIPSE.

BY PROF. W. H. ECHOLS.

If a circle rolls without slipping inside of a fixed circle of double the radius whose center is O, then in the plane of the fixed circle the center of the rolling circle describes an equal circle whose center is O, any point on the circumference of the rolling circle describes a straight line through O, and any other point in the plane of the rolling circle describes an ellipse.

This kinematical truth leads to the practical method of turning ellipses, and also to valuable geometrical methods of drawing the curve.



1. Let C be a position of the center of the rolling circle and P a point moving with it describing an ellipse. Draw in

the plane of the moving circle and through its center C the secant PC cutting the circumference in A and B. The points A and B move in straight lines through O perpendicular to each other. From the nature of the curve these lines are known to be the axes whose semi-lengths are PA and PB. Thus the familiar method of drawing the curve when the axes are given.

2. M is the point of contact of the rolling circle with the base circle, it is therefore the instantaneous center about which all points in the plane of the rolling circle are rotating. Hence MP is normal to the path of P at P, and therefore normal to the diameter conjugate to OP. Thus OF is the direction of that diameter and $OQ = PM$ is its semi-length; for $PM \times PF = PA \times PB$, being secants to the rolling circle from P, and $PF = PO \sin POQ$. Hence $PA \times PB = PO \times QO \sin POQ$ and OQ must be the semi-diameter conjugate to OP.

Given a pair of semi-conjugate diameters in position to draw the curve. From the extremity P of one of them draw P M normal to the other and equal to it in length, cutting it in F. As the line of constant length P M moves so that the fixed point F moves on the diameter O Q and the point M on the straight line M O through the center, the point P describes the ellipse.

This is a simple and easy way of drawing the projections of circles. I have not seen this method given, nor do I know of its being used.*

3. Minchin in his Uniplanar Kinematics gives a triangle, two of whose vertices move on the conjugate diameters while the third describes the ellipse. The proof there given is analytical. There are really two such triangles, which are shown in the figure as R P F and R′ P F.

R and F being points on the circumference of the moving circle, move on the diameters P O and Q O as P describes the ellipse. The triangle is determined as follows: P F is the distance of P from O Q and P R the distance of Q from O P, the angle at P is $90° - POQ$. Join R M, P R M is right angled at R and $< PMR = < POQ$. Hence $PR = PM \sin POQ$.

Also as R′ and F move in O P and O Q respectively, P traces the curve and $P R' = P R$, also $<F P R' = 90^\circ + P O Q$.

What has gone before applies equally to the rolling circle system whose center is C', if the proper letters be accented.

4. The figure shows that no *triangle* can move with a vertex on each *axis* of an ellipse and the third vertex trace the curve, since the above triangles (3) degenerate into one straight line when the pair of conjugates become the axes. Also that no *straight* line can move with two of its fixed points on a pair of conjugates while a third fixed point on it traces the curve.

5. In general any secant cuts the circle in two points, the intercepted portion of which may be taken for the base of a triangle whose extremities describe straight lines through O as the vertex at P describes the curve. If the secant passes through *C* the ends of the base move in straight lines normal to each other, If the secant passes through P the triangle becomes a straight line.

Since C traces a circle, any straight line from C to a point on the circumference may be taken as the base of a triangle one end of which moves in the circular path of C, the other in a straight line through O while the vertex at P traces the ellipse. This triangle may become either of the straight lines P B C or P C A, or with the accents.

Again any secant P B C or P C A moving so that C moves on the circle described by C, while B slides on O B, or A on O A, describes the ellipse with P. That is any vertex O of an isotriangle O C B remaining fixed in position and the odd side O B in direction, then as B moves on O B every point on C B describes an ellipse. A Paucellier Motion attached to B then gives the linkage for describing ellipses without a sliding joint. These are all familiar motions in Kinematics.

6. Any secant drawn to the circle (C) from P will have two segments, either of which is the length of a semi-diameter, the other segment is the distance of the tangent at the extremity of its conjugate from the center of the curve.

Any two secants drawn from P to the extremities of a diameter through C, are a pair of semi-conjugate diameters of the ellipse in length. The angle between them is the compliment of the angle between these diameters in the ellipse. The angles which each makes with the axes of the curve are at once determined as in the figure.

7. This circle gives a number of easy constructions relating to the ellipse.

If any two points A and B in a plane describe straight lines Ox and Oy respectively, any third point in the plane describes an ellipse which is at once deteɹmined by drawing the circle through O, A and B. Either the diameters or a conjugate to OP can be drawn at once.

Also a simple construction for drawing a pair of conjugate diameters which shall include a given angle, the equi-conjugate diameters, &c., &c.

The figure also shows that the locus of a point on the normal at a distance from the point of contact equal to the semi-diameter conjugate to that through the point of contact is the concentric circle with radius $a \pm b$.

Huntsville, Ala., July, 1889.

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*At the time this paper was written I was under the impression that this construction was new. I had been using it for a long time for drawing the projections of circles without going to the trouble of first constructing the axes and had made rather a careful search among English works to see it if it could be found. Not being able to find it even in Eagle's Constructive Geometry of Plane Curves (MacMillan 1885), the most elaborate thing of its kind I know of in English. I finally wrote this paper with the purpose in view of placing before draughtsmen this very elegant and simple little way of drawing the curve on the conjugates. Its extreme simplicity made me hesitate to put forth the claim of newness, as I felt sure it could only be through my own ignorance that I had failed to find it in print. After some months hesitation I sent it to the *Annals of Mathematics* for publication, but fortunately discovered the author in time to withdraw the paper.

The construction is due to Manheim and may be found in his Elements de la Geometrie Cinematique, page 164. In a foot note to the construction there given he says, "C'est en 1857, dans les *Nouvelles Annales de Mathematiques*, p. 188, que j'ai donne pour la premiere fois cette construction." While this remark applies to the construction of the axes given a pair of conjugates, that construction in the text includes the above.

The only apology I can now make for presenting this construction here is its rarity or total absence from English text-books. W. H. E.

ON THE ESTABLISHMENT OF THE TRUE MERIDIAN BY MEANS OF OBSERVATIONS ON THE SUN WITH THE ENGINEER'S SOLAR INSTRUMENT.

BY GEO. R. DEAN.

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1. After a somewhat extended examination the writer has observed that all engineering text-books, which treat the subject at all, give an inadequate exposition of the principles on which the use of the Engineer's Solar is based. All agree that the instrument should not be used near noon, and offer as the reason for such instruction, that the error in azimuth of the line of sight corresponding to an error in setting the Sun's declination, is greater at this time than at any other. While this is true, there is another and if possible more potent reason why the instrument should not be used near noon. For as will be shown in the sequel, should the latitude and declination be set off with theoretical exactness, the liability to error still exists and the possibility of establishing a false meridian is highly probable.

2. The Solar:—The instrument consists (mathematically) of a vertical axis to which is hung a terrestrial line of sight capable of motion in altitude and azimuth, and supplied with clamps for arresting each of these motions.

Rigidly attached to the terrestrial line of sight, in the same vertical plane with it and normal to it, is a polar axis, about which revolves freely a celestial line of sight capable of being set at any angle with the polar axis.

3. The principle underlying the use of the Solar:—If at any point of the Earth whose latitude is known, the instrument be planted, the vertical axis made vertical, the polar axis set at an angle with the horizon equai to the latitude of the place, and the celestial line of sight set at an angle with the polar axis equal to the polar distance of the Sun at the instant of observation:

then, if the celestial line of sight be brought upon the Sun (by rotating the instrument about its vertical axis and the celestial line of sight about the polar axis), the terrestrial line of sight must be in one of *two* vertical planes. One of these planes is the True Meridian, the other a *False* Meridian. Measuring azimuths (for our present purpose) from the north point around through the East: the azimuth of the False Meridian will be twice the azimuth of the Sun.

This is simply shown as follows:

4. As the polar axis revolves about the vertical it generates the surface of a cone of revolution whose semi-vertical angle is the co-latitude of the place; therefore this surface passes through the Celestial Pole. If now the celestial line of sight be brought upon the Sun, the polar axis must be found in a cone of revolution whose axis is the celestial line of sight and whose semi-vertical angle is the polar distance of the Sun; therefore this surface passes through the Celestial Pole. These two cones having their vertices common, have two elements in common, one of which passes through the Celestial Pole. It is easy to see (from elementary geometry) that the vertical plane through the Sun bisects the diedral angle between the two common elements. Therefore the azimuth of the second element is twice that of the Sun. Fig. (2) represents the Celestial Sphere projected in plan and elevation upon the Horizon and True Meridian respectively. The traces with it of the cones in which the polar axis is found, are projected horizontally in the ellipse whose minor axis is in the plan of the celestial line of sight C′ S′, and the circle whose center is Z′ and radius Z′ P′, respectively. They intersect in points P′ and P′ equidistant from C′ S′.

5. In the spherical triangles Z P S and Z P′ S (Fig. 1), the three sides of the one are respectively equal to the three sides of the other; hence the triangles are either equal or symmetrical, i. e., P′ coincides with P, or the azimuth of P′ is twice that of S.

Again; from the same figure, where the angles P Z S and P′ Z S are A and A′, respectively:

$\cos SP = \cos SZ \cos ZP + \sin SZ \sin ZP \cos A \;\ldots\ldots$ (1);

$\cos SP = \cos SZ \cos ZP' + \sin SZ \sin ZP' \cos(A - A') \ldots$ (2).

Since $SP = SP'$ and $SP = ZP'$; $\cos(A - A') = \cos A$ and therefore $A' = 0$ or $2A$.

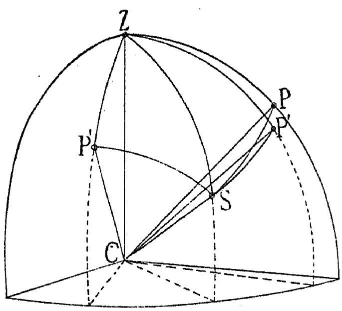

FIG. I

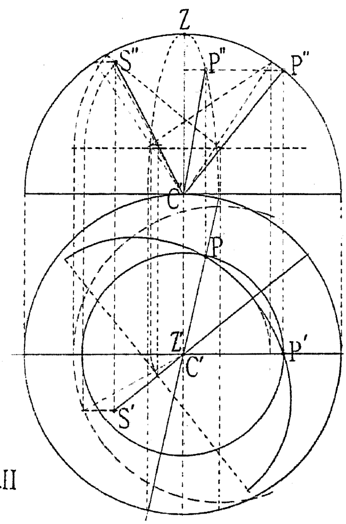

FIG. II

6. At sunrise and sunset or near these times, the celestial line of sight may be brought to bear upon the Sun while the polar axis and terrestrial line of sight are in a vertical plane quite near the True Meridian, which we call the False Meridian. There is no possibility of confounding the two, since, when the polar axis is in the False Meridian it points south. Therefore this case may be dismissed since the error of mistaking the meridian is provided against by merely a general knowledge of the points of the compass. When however the Sun is near the Meridian, the False Meridian approaches the True Meridian (and the polar axis points north) very nearly, coinciding with it exactly when the Sun is on the Meridian.

Hence when the Sun is nearly on the Meridian, it becomes impossible to distinguish between the True and False Meridian even with the magnetic needle as a guide. The text-books guard against this mistake by instructing that the polar axis be set in the Meridian as nearly as may be by the needle and forbidding the use of the instrument within one hour of noon. They all agree however in attributing the reason for this instruction solely to the relatively great error in alignment due to any error in setting off the angles, when the instrument is used at this time.

7. In order to investigate the error in locating the Meridian when the instrument is used near noon, consider the Z P S triangle, and let λ, δ, α, Z and H, be the latitude of the place, the polar distance of the Sun, its altitude, azimuth, and hour angle respectively; then,

$$\sin\delta = \sin\alpha\sin\lambda + \cos\alpha\cos\lambda\cos Z \quad . \; . \; . \; . \; . \quad \text{(I)}.$$

If this be differentiated, first considering α and λ as constant, and then α and δ as constant we get the following formulæ, after an easy reduction by use of fundamental relations in the Z P S triangle:

$$dZ_{\delta} = -\frac{d\delta}{\cos\lambda\sin H} \quad . \; . \; . \; . \; . \quad \text{(II)},$$

$$dZ_{\lambda} = -\frac{d\lambda}{\cos\lambda\tan H} \quad . \; . \; . \; . \quad \text{(III)}.$$

These are the formulæ given in Johnson's Theory and Practice of Surveying, for the errors in locating the Meridian due to an error in setting off declination and latitude, respectively.

These formulæ are only approximately true when $d\delta$, $d\lambda$ and dZ are small, and they fail altogether when the Sun is about to transit or when H is small, since they make the error infinitely great whenever $d\delta$ or $d\lambda$ is finite. These formulæ, and therefore any tables calculated by them give an entirely erroneous idea of the Meridian error near noon. In point of fact, in order to get a clear conception of the Meridian error we must go back to the finite difference equation itself, and must not proceed to the limit until we are sure of getting the desired approximation.

8. Consider a point P′ on the same almucantar with P, and in the Z P′ S triangle let the angle S Z P′ be Z′, and side S P′ be δ' : then we have

$$\cos\delta = \sin\alpha \sin\lambda + \cos\alpha \cos\lambda \cos Z, \text{ and}$$

$$\cos\delta' = \sin\alpha \sin\lambda + \cos\alpha \cos\lambda \cos Z', \quad \ldots\ldots \quad \text{(IV)};$$

$$\text{or } \cos\delta - \cos\delta' = \cos\alpha \cos\lambda\,(\cos Z - \cos Z');$$

whence, if $\delta - \delta' = \Delta\delta$; $Z - Z' = \Delta Z$;

$$\sin\,(\delta - \tfrac{1}{2}\Delta\delta) \sin\tfrac{1}{2}\Delta\delta = \cos\alpha \cos \cos\lambda \sin\,(Z - \tfrac{1}{2}\Delta Z) \sin\tfrac{1}{2}\Delta Z \quad \ldots\ldots \quad \text{(V)}.$$

If in this equation we pass to the limit as $\Delta\delta$ and ΔZ become $d\delta$ and dZ we get equation II as above, but in passing to the limit in $\sin\,(Z - \frac{1}{2}\triangle Z)$ we let $\triangle Z$ go out in comparison with Z which cannot be done when Z itself is infinitely small or zero, as is the case when the sun is on the Meridian. Hence the derivative only gives an approximate formula when Z is large compared with $\triangle Z$, which is usually the case.

When the Sun is on the Meridian or Z is π, then the meridian error corresponding to the declination error is

$$\sin\tfrac{1}{2}\Delta Z = \sqrt{\frac{\sin\,(\delta - \frac{1}{2}\Delta\delta) \sin\Delta\frac{1}{2}\delta}{\cos\alpha \cos\lambda}} \quad \ldots\ldots \quad \text{(VI)}.$$

This shows that if $\Delta\delta$ is negative $\triangle Z$ is impossible, or that with the Sun on the Meridian and a polar distance be set off greater than the true polar distance it is impossible to bring the celestial line of sight on the Sun. When $\Delta\delta$ has two equal finite values with opposite sign. Thus for $\lambda = 40^\circ$; $\delta = 105^\circ$; $\alpha = 35^\circ$; we have, if $\Delta\delta = 01'$, $\Delta Z = 1^\circ\,42'\,54''$.

To compute this error for a small hour angle we may use the formulæ,

$$\sin Z = \frac{2}{\cos\alpha \cos\lambda}\sqrt{\sin S \cos\,(S+\alpha) \sin\,(S-\delta) \cos\,(S+\lambda)},$$

$$\sin Z' = \frac{2}{\cos \alpha \cos \lambda}\sqrt{\sin S' \sin (S'+\alpha) \sin (S'-\delta') \cos (S'-\lambda)};$$

where S and S′ are the half sum of the sides in the triangles Z P S and Z P′ S respectively, and where

$$\alpha = \text{arc-}\sin \left\{ \frac{\sin \delta}{\cos \theta} \sin (\lambda + \theta) \right\};$$

θ being arc-tan ($\tan \delta \cos H$).

In the hour angle one minute of time is equivalent to 15 minutes of arc. The errors in azimuth of line of sight Meridian due to an error of one minute of arc in declination, for $\lambda = 40^\circ$, $\delta = 105^\circ$, when the Sun is 10, 20, 30, and 60 minutes in time, from the Meridian, have been computed by the above formulæ and are respectively,

H =	0^m	10^m	20^m	30^m	60^m	2^{hrs}
Z =	0°	$2^\circ 55' 10''$	$5^\circ 40' 31''$	$8^\circ 49' 30''$	$17^\circ 21' 8''$	$32^\circ 13'$
ΔZ =	$1^\circ 42' 54''$	$17' 58''$	$11' 40''$	$9' 15''$	$5' 12''$	$2' 38'$

Johnson's Surveying gives as computed by (II):

H =	0^m	10^m	20^m	30^m	60^m	2^{hrs}
ΔZ =	∞	$29' 30''$	15	$10'$	$5' 03''$	$2' 37''$

These results show that the instrument can not be used any nearer noon than the text-books instruct, but the error is not indeterminate, and is not infinite at noon as the derivative might lead us to believe.

8. To compute the error in azimuth due to an error in latitude, we write down as before,

$$\cos \delta = \sin \alpha \sin \lambda + \cos \alpha \cos \lambda \cos Z.$$

$$\text{and } \cos \delta = \sin \alpha \sin \lambda' + \cos \alpha \cos \lambda' \cos Z':$$

whence follows, since $\lambda - \lambda' = \Delta\lambda$. etc.;

$$2 \tan \alpha \sin \tfrac{1}{2} \Delta\lambda \cos (\lambda - \tfrac{1}{2} \delta\lambda) = \cos \lambda \cos Z - \cos (\lambda - \Delta\lambda) \cos Z',$$

$$\text{or } \tan \alpha \sin \tfrac{1}{2} \Delta\lambda = \sin (Z - \tfrac{1}{2} \Delta Z) \sin \tfrac{1}{2} \Delta Z;$$

since $\Delta\lambda$ is always small compared with λ.

To get the error for the Sun on the Meridian, Z = 0; hence

$$\sin \tfrac{1}{2} \Delta Z = \sqrt{\tan \alpha \sin \tfrac{1}{2} \Delta \lambda};$$

from which if $\Delta\lambda$ is negative. or $\lambda' > \lambda$, it is impossible to bring the line of sight on the Sun. If $\Delta\lambda$ is positive and equal to 01′, then $\Delta Z = 1^\circ 9' 22''$ for $\alpha = 35^\circ$.

In order to compute these errors for other times than near noon, we must use

$$\sin Z = \frac{2}{\cos \alpha \cos \lambda}\sqrt{\sin S \cos (S + \alpha) \sin (S - \delta) \cos (S + \lambda)},$$

$$\sin Z' = \frac{2}{\cos \alpha \cos \lambda'}\sqrt{\sin S' \cos (S' + \alpha) \sin (S' - \delta) \cos (S' + \lambda')}.$$

The results agree with those given in Johnson's Surveying, computed by (III,) to within 20 minutes of noon.

Thus the error in establishing the Meridian when the Sun is about to transit, due to any error in setting the declination or latitude is so great as to forbid the successful use of the instrument near this time, and which therefore excludes the danger of establishing the False Meridian.

The above is an extract from a thesis on the Solar by the writer.

EXERCISES.

I.

Two vertices A and B of a triangle $A\ B\ C$ describe straight lines which meet at the angle ω; show that the area of the curve described in their plane by the vertex C is

$$\Omega = \frac{\pi}{2}\,(a^2 + b^2 + c^2 - 4\triangle \operatorname{ct} \omega).$$

$\triangle$ being the area of the triangle $A\,B\,C$. [*W. H. Echols.*]

2.

In exercise 1, find the whole length of the envelope of the side c. [*W. H. Echols.*]

3.

Two parallel straight lines are distant apart d; it is required to unite them by circular arcs which shall have between them a common tangent of length t. [*Elmo G. Harris.*]

4.*

The angles of depression of two towns T and T', n miles apart, are observed from a balloon and found to be *arc-cot* a and *arc-cot* a', respectively; the balloon moves in a line whose azimuth with respect to the line joining the two towns is *arc-cos* θ; upon arriving at a point known to be m miles (horizontally) from the first point of observation the angles of depression of T and T' are now observed to be *arc-cot* b and *arc-cot* b' respectively. What was the height of the balloon at each station? [*Geo. R. Dean.*]

5.

Two straight lines $O\,P$ and $O\,Q$ are of lengths b' and a' respectively. From P a perpendicular $P\,M$ is drawn to $O\,Q$ and equal to it, cutting it in N. Show that the equation to the locus of P, as the point N moves on $O\,Q$ and the point M on $O\,M$, referred to $O\,Q$ and $O\,P$ as axes of x and y respectively, is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$ [*W. H. Echols.*]

6.

Regarding the portion of the tangent to the hyperbola intercepted by the asymptotes as one diagonal of a square, what are the loci of the extremities of its other diagonal? [*W. H. Echols.*]

*A generalization of an exercise in Snowball's Trigonometry.

ERRATA.

Page 9, line 7 from bottom, *for* d^1 *read* d_1.

Page 18, line 6, *for* $\frac{\text{si}\,\varphi\, d\varphi}{\sqrt{1\, m\, \varphi}}$ *read* $\frac{\text{si}\,\varphi\, d\varphi}{\sqrt{2\, m\, \varphi}}$.

Page 19, line 10, *for* co $(\frac{1}{2}\, m\, s^2\, ds$ *read* co $(\frac{1}{2}\, m\, s^2)\, ds$.

Page 22, line 21, *dele* A. C.

Page 23, line 7, *dele* A. C.

Page 32, line 7, *for* "what is known as a member of" *read* "a member of what is known as."

Page 40, line 4 from bottom, *for* Ripper *read* Roepper.

CONTENTS.

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