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IMAGINARIES IN ANALYTIC GEOMETRY.

BY DR. WILLIAM B. SMITH, COLUMBIA, MO.

In Arts. 253—261 and elsewhere in the classic treatise of Salmon on *Conic Sections*, as well as in other works of like scope, will be found a number of analytic properties of the binary quadric, or General Equation of second degree in Cartesian resp. trilinear coördinates, interpreted geometrically in terms of imaginary points and right lines, whether in finity or at infinity. The formal correctness of these interpretations is, of course, not to be questioned, but it is equally manifest that the visible geometric depiction is altogether inadequate to express the relations under consideration.

By use of the quadrantal versor i as an operator, to denote the turning of an ordinate y through a right angle into perpendicularity to the plane of X Y. Mr. Carr, in his *Synopsis of Pure Mathematics*, enlarges measurably the range of geometric representation. Thus the Equation

$$x^2+y^2=a^2,$$

for $$-a \leqq x \leqq a,$$

is depicted by a circle of radius a about the origin, the axes being rectangular. For x lying outside of these extremes the value of y is $i\sqrt{x^2-a^2}$, and the geometric picture is accordingly an equiaxal Hyperbola having the same parameter and real axis as the circle, but in a plane normal to X Y along X. If now, for convenience, this Hyperbola be rotated about X into the original plane X Y, whereby the foot of the ordinate y will not be changed, in this new position it is called, following Poncelet, *supplementary* to the circle as *principal.* In general the two curves

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

are principal and supplementary. Manifestly, supplementaries to the same principal will vary with the choice of conjugate diameters for coördinate axes.

With help of the supplementary curve, many puzzling properties of conics, such as

All circles meet in the same two imaginary points at infinity;

Concentric circles touch in four imaginary points at infinity;

All confocal conics have four common tangents imaginary and determining four foci, two real, two imaginary, as two pairs of their opposite intersections;

and the like, may now be interpreted geometrically and visibly.

Two reflections, however, suggest themselves. By turning the perpendicular curve into the plane X Y and then using its properties to supplement the properties of the principal, we seem really to surrender the problem of interpreting our analytic properties through the circle and to say in effect. "We can not understand these of the visible circle, but we may understand them of the visible hyperbola." Accordingly, the problem of rendering these properties, when affirmed of the circle, intelligible to intuition, seems scarcely to have been met and solved but rather evaded. The burden which proved too heavy for the principal shoulder has been shifted over to the supplementary one. Thus, when asked to say in what sense the circle

$$x^2+y^2=a^2$$

has a pair of imaginary asymptotes

$$x^2+y^2=0,$$

we make answer that the supplementary hyperbola

$$x^2+y^2=a^2$$

has a pair of real ones

$$x^2-y^2=0.$$

The answer is indeed quite correct, but not quite relevant.

Again, in dealing with extra-real values of the coördinates, either of two ways seems logically open: to admit all or to admit none. Choosing the latter, we must say of the Asymptotes

$$x^2+y^2=0$$

simply that they *are not*, no finite real values satisfying their Equation; this latter is accordingly a mere analytic symbolism void of geometric content. This answer is entirely correct and consistent, involving no internal contradiction. So with respect to imaginary points of intersection of conics, we may say curtly there *are no such points* and so end the discussion. But if we choose the other path and admit any imaginary values to equal rights with real ones, then we must admit all, "for there is no difference." Any reason which legitimates the value i for y in $x^2+y^2=1$ must legitimate the same value for x and the general value $a+ib$ for both. The fact is, so soon as the ditensive unit i is recognized at all the domain of number becomes a manifold doubly extended and is no longer to be pictured by a continuity of points along a single axis as X or Y, but requires a surface, as a plane, for its complete depiction. Very naturally, then, the geometric interpretation of the Equation in x and y, where each may be of the form $a+ib$, as a curve in the plane X Y, while *quoad* perfect, is yet incomplete, for there is no place on either axis for the geometric picture of an imaginary value, and hence no place in their plane for the picture of a pair of such values. Evidently, then, if we would interpret the Equation completely, we must construct the values of the coördinates completely: we must assume two planes, instead of two axes, of x and y. In each of these planes we assume two axes at right angles, the one of pure reals, the other of pure imaginaries. For convenience, suppose the planes perpendicular to each other, then in general we shall have four mutually perpendicular right lines, which are possible only in at least four-fold space. Such a space, though perfectly reasonable, is not imaginable, our intuition reaching only to three dimensions. Our equation constructed in this space would yield a solid as a border between two four-fold extent, and while amenable to analytic treatment would still defy envisagement as effectually as did our imaginary elements in the original plane. However, there is nothing to prevent our assuming two axes of reals at right angles, and a third axis normal to their plan e at their intersection as the *common axis* of pure imaginaries. If this be named the Z-axis, then the whole domain of value of x will be geometrically the X Z-plane, and of y, the Y Z-plane. Now put

$$x=u+i\,u' \qquad \text{and} \qquad y=v+i\,v';$$

then these two points (u, u'), (v, v'), in X Z and Y Z, are two opposite vertices of a parallelogram, of which the origin and the point (x, y) are the other pair of vertices. The rectangular coördinates of this point (x, y) are plainly u, v, $u'+v'$, or u, v, z, all of which are always real. To any pair (x, y) corresponds a triplet (u, v, z); accordingly the complete depiction of the equation in x and y, when complex values are admitted, will be the perfect depiction of the corresponding equation in u, v, z when only real values are admitted. This latter will of course be a surface in the space of u, v, z. It remains to transform the Equation in x, y into an equation in u, v, z.

The General Equation of first degree in x and y is

$$lx+my+n=0; \tag{1}$$

whence
$$l(u+iu')+m(v+iv')+n=0; \tag{2}$$

whence
$$lu+mv+n=0, \tag{3}$$

and
$$lu'+mv'=0 \tag{4}$$

Equation (3), independent of z, is that of a plane parallel to Z, its trace on the plane U V being the right line

$$lu+mv+n=0.$$

Equation (4) does not limit in any way the total locus of (3), but merely declares how the z of each point of the locus is set together out of u' and v'. Here the constants l, m, n have been supposed real, as is uniformly done in discussions of this equation. But that supposition is by no means a necessary one. If we attribute to them the most general values,

$$a+ia', \quad b+ib', \quad c+ic',$$

then result the equations

$$\begin{aligned} au+bv+c-a'u'-b'v'&=0 \\ a'u+b'v+c'+au'+bv'&=0 \qquad : \qquad \text{also,} \\ z-u'-v'&=0. \end{aligned}$$

whence, eliminating u' and v', we have

$$\begin{vmatrix} au+bv+c, & -a', & -b', \\ a'u+b'v+c', & a, & b, \\ z, & -1, & -1, \end{vmatrix} = 0, \quad \text{or}$$

$$(a^2-ab+a'^2-a'b')u-(b^2-ab+b'^2-a'b')v+(a'b-ab')z \\ +ac-bc+a'c'-b'c'=0,$$

the equation of a plane not in general perpendicular to the original plane U V. Examples of such oblique planes will hereafter present themselves. Omitting at this point further discussion of this equation, let us pass to Equations of second degree. Of these the simplest is that of the circle about the origin,

$$x^2+y^2=a^2.$$

Or $$(u+i\,u')^2+(v+i\,v')^2=a^2; \qquad (5)$$

whence $$u^2+v^2-(u'^2+v'^2)=a^2, \qquad (6)$$

and $$u\,u'+v\,v'=0. \qquad (7)$$

From (7) and $u'+v'=z$ there result

$$v'=\frac{u\,z}{u-v}, \qquad u'=-\frac{v\,z}{u-v}, \qquad (8)$$

whence, on substitution in (6) and reduction,

$$(u^2+v^2)\,[(u-v)^2-z^2]-a^2(u-v)^2=0, \qquad (9)$$

a surface of fourth degree.

To get a clearer idea of this quartic let us introduce polar coördinates by the relations

$$u=\rho\,\underline{\vartheta}, \quad v=\rho\,\vartheta_{|},$$

where $\underline{\vartheta}$ and $\vartheta_{|}$ stand for cosine and sine of ϑ, the inclination of ρ from U-axis. Then (9) becomes, on rejecting ρ^2,

$$(\rho^2-a^2)\,(\underline{\vartheta}-\vartheta_{|})^2-z^2=0. \qquad (10)$$

Here $(\underline{\vartheta}-\vartheta_{|})^2$ is a pure number positive and constant for ϑ constant; call it k^2; then, taking ρ and z as coördinates of the curve of section of the surface with the plane through Z sloped ϑ to U, we have, as its rectangular equation,

$$\frac{\rho^2}{a^2}-\frac{z^2}{a^2\,k^2}=1, \qquad (11)$$

an Hyperbola, or, for varying ϑ, a family of Hyperbolas. The *parameter of this family* is k^2; the real axes are all $2a$ and form the pencil of diameters of the circle

$$u^2+v^2=a^2;$$

for $\vartheta = 0$ the conjugate axis is $2a$, the hyperbola is equilateral; as ϑ increases to $\frac{1}{4}\pi$ the conjugate axis shrinks to 0, the hyperbola flattens to a doubly laid right line bisecting outside of the circle the angle U O V; as ϑ goes on increasing to $\frac{1}{2}\pi$, k^2 passes through the same system of values, yielding the same system of hyperbolas, in opposite order; for ϑ increasing to $\frac{3}{4}\pi$, k^2 rises to its maximum, 2, thence for ϑ increasing to π it once more sinks through opposite stages to its original value, 1. Herewith the circuit of its values is complete and is merely repeated as ϑ passes from π to 2π. Accordingly, as a plane turns about z it cuts the surface continually in an Hyperbola, with vertex on the circle

$$u^2 + v^2 = a^2,$$

with constant real axis $2a$, and with conjugate axis ranging continuously from 0 for $\vartheta = \frac{1}{4}\pi$ or $\frac{5}{4}\pi$ to $2a\sqrt{2}$ for $\vartheta = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$. The surface is symmetric with respect to two planes bisecting the angles of the real axes, U and V, as becomes analytically clear on turning the axes through an angle, $\frac{1}{4}\pi$; it consists of two halves compendent along the inner bisector, $u = v$.

Now suppose $a = 0$; the circle reduces to the point-circle

$$x^2 + y^2 = 0,$$

which is also the pair of imaginary Asymptotes

$$(x + iy)(x - iy) = 0.$$

But $a = 0$ reduces our equation (9) to

$$u^2 + v^2 = 0 \quad \text{or} \quad (u - v)^2 - z^2 = 0. \tag{12}$$

Of these the first is pictured completely by the origin, since u and v are expressly real, the second breaks up into the two disjunctive equations

$$u - v - z = 0 \quad \text{and} \quad u - v + z = 0. \tag{13}$$

These are depicted by two planes meeting on the right line $u-v=0$, bisecting the coördinate angle U O V; the first also bisects the angle V O Z; *i. e.*, the planes bisect the coördinate angle U O V of the *original* axes and are inclined, each at an angle whose tangent is z, to the plane of those axes. Also, the plane turning about z cuts this pair of planes in the curve

$$k^2 \rho^2-z^2=0, \tag{14}$$

i. e., in the pair of right lines

$$k\rho-z=0 \quad \text{and} \quad k\rho+z=0. \tag{15}$$

But these right lines are plainly the Asymptotes to the section of the surface made by the rotating plane, namely, to the Hyperbola

$$k^2 \rho^2-z^2=k^2 a^2. \tag{16}$$

Hence we see that the locus

$$x^2+y^2=0$$

is genuinely and completely asymptotic to the locus

$$x^2+y^2=a^2,$$

every rectilinear radial section of the first being asymptotic to the corresponding hyperbolic section of the second.

Now change the sign of a^2; then results the purely imaginary circle

$$x^2+y^2=-a^2.$$

Its complete spatial depiction is obtained at once by changing the sign of a^2 in the foregoing reasoning; the asymptotic planes are unaffected; while all the hyperbolas pass over into their conjugates. Thus the imaginary circle stands to the real circle, not only analytically but also visually, precisely as the conjugate hyperbola stands to its primary, the one being quite as "real" as the other, and both having the common real asymptotic planes

$$x^2+y^2=0.$$

The complete spatial depiction of the real and (so-called) imaginary ellipses,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm 1,$$

is now easily apparent. It suffices to supplace y with $\frac{b}{a}y$ in in the foregoing. The general relations remain undisturbed.

Let us now pass to the rectangular hyperbola,

$$x^2-y^2=a^2 \tag{17}$$

Proceeding as in case of the circle we obtain

$$(u^2-v^2)\,[(u+v)^2+z^2]-a^2(u+v)^2=0, \tag{18}$$

or

$$(\vartheta+\vartheta_1)^2\rho^2+z^2=a^2(\vartheta+\vartheta_1)^2(\vartheta^2-\vartheta_1^2), \tag{19}$$

an Ellipse in the plane through Z turned ϑ from U, with center at origin, one axis $2a/\sqrt{\vartheta^2-\vartheta_1^2}$, the other $2a(\vartheta+\vartheta_1)/\sqrt{\vartheta^2-\vartheta_1^2}$. For $\vartheta=0$ this ellipse becomes a circle with diameter $2a$; as ϑ increases both axes increase, the second the faster, which is therefore the axis major, until for $\vartheta=\frac{1}{4}\pi$ both become infinite. For ϑ ranging from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$ the sections are strictly imaginary ellipses, since both ρ and z are expressly real; *i. e.*, no part of the real surface lies in this quadrantal region. As ϑ increases from $\frac{3}{4}\pi$ to π, the section, once more real, shrinks from an infinite ellipse to the initial circle, radius a; and herewith the circuit of values is complete, to be retraced as ϑ ranges from π to 2π. The minor axes of all these real elliptic sections are the primary diameters of the hyperbola under consideration. The Asymptotes

$$x^2-y^2=0=(x-y)\,(x+y)$$

are two planes through Z bisecting outerly and innerly the original coördinate angles and touching the surface at infinity all around along the infinite ellipse.

Now change the sign of a^2; the hyperbola passes over into its conjugate

$$x^2 - y^2 = -a^2;$$

all the elliptic sections which were real for a^2+, namely, all for ϑ ranging from $-\frac{1}{4}\pi$ to $+\frac{1}{4}\pi$, now become imaginary, while all which were imaginary, namely, for ϑ ranging from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$, now become real; and the Asymptotes remain the same.

It is hardly necessary to detain the reader with further exemplifications. It seems entirely evident that the so-called imaginary points, lines, circles, ellipses, yea, curves and properties in general, are no longer imaginary in the *lucus a non lucendo* sense of *un*imaginable, but that they exist for the spatial imagination altogether as genuinely as any of the reals of Analytic Geometry.

Further discussion is reserved for the present.

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OUR BELIEF IN AXIOMS, AND THE NEW SPACES.

BY DR. GEORGE BRUCE HALSTED, UNIVERSITY OF TEXAS.

"Prove all things, hold fast that which is good," does not mean demonstrate everything. From nothing assumed, nothing can be proved. "Geometry without axioms," was a book which went through several editions, and still has historical value. But now a volume with such a title would, without opening it, be set down as simply the work of a paradoxer.

The set of axioms far the most influential in the intellectual history of the world was put together in Egypt: but really it owed nothing to the Egyptian race, drew nothing from the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that Geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoön.

In a geometry occur the most noted set of axioms, the geometry of Euclid, a pure Greek professor at the University of Alexandria.

Not only at its very birth did this typical product of the Greek genius assume sway as ruler in the pure sciences, not only does its first efflloresence carry us through the splendid days of Theon and Hypatia, but unlike the latter, fanatics cannot murder it; that dismal flood, the dark ages, cannot drown it. Like the phœnix of its native Egypt, it rises with the new birth of culture. An Anglo-Saxon, Adelard of Bath, finds it clothed in Arabic vestments in the land of the Alhambra. Then clothed in Latin, it and the new-born printing press confer honor on each other. Finally back again in its original Greek, it is published first in Queenly Venice, then in stately Oxford, since then everywhere. The latest edition in Greek is just issuing from Leipsic's learned presses.

How the first translation into our cut-and-thrust, survival-of-the-fittest English was made from the Greek and Latin by Henricus Billingsly, Lord Mayor of London, and published with a preface by Jonh Dee the Magician, may be studied in the Library of our own Princeton College where they have, by some strange chance, Billingsly's own copy of the Latin version of Commandine bound with the Editio Princeps in Greek and enriched with his autograph emendations. Even to-day in the vast system of examinations set by Cambridge, Oxford, and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms.

The American ideal is success. In twenty years the American maker expects to be improved upon, superseded. The Greek ideal was perfection. The Greek Epic and Lyric poets, the Greek sculptors, remain unmatched. The axioms of the Greek geometer remained unquestioned for twenty centuries.

How and where doubt came to look toward them is of no ordinary interest, for this doubt was epoch making in the history of mind.

Among Euclid's axioms was one differing from the others in prolixity, whose place fluctuates in the manuscripts, and which is not used in Euclid's first twenty-seven propositions. Moreover it is only then brought in to prove the inverse of one of these already demonstrated.

All this suggested, at Europe's renaissance, not a doubt of the axiom, but the possibility of getting along without it, of deducting it from the other axioms and the twenty-seven propositions already proved. Euclid demonstrates things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third. Yet when he has perfectly proved that lines making with a transversal equal alternate angles are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy postulate or axiom;

"If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles."

Do you wonder that succeeding geometers wished by demonstration to push this unwieldy thing from the set of fundamental axioms.

Numerous and desperate were the attempts to deduce it from reasonings about the nature of the straight line and plane angle. In the "Encyclopœdie der Wissenschaften und Künste; Von Ersch und Gruber;" Leipzig, 1838; under "Parallel," Sohncke says that in mathematics there is nothing over which so much has been spoken, written, and striven, as over the theory of parallels, and all, so far, (up to his time) without reaching a definite result and decision.

Some acknowledged defeat by taking a new definition of parallels, as for example the stupid one, "Parallel lines are everywhere equally distant," still given on page 33 of Schuyler's Geometry, which that author, like many of his unfortunate prototypes, then attempts to identify with Euclid's definition by pseudo-reasoning which tacitly assumes Euclid's postulate, e. g. he says p. 35; "For, if not parallel, they are not everywhere equally distant; and since they lie in the same plane, must approach when produced one way or the other; and since straight lines continue in the same direction, must continue to approach if produced farther; and if sufficiently produced, must meet." This is nothing but Euclid's assumption, diseased and contaminated by the introduction of the indefinite term "direction."

How much better to have followed the third class of his predecessors who honestly assume a new axiom differing from Euclid's in form if not in essence. Of these the best is that called Playfairs; "Two lines which intersect cannot both be parallel to the same line."

The German article mentioned is followed by a carefully prepared list of ninety-two authors on the subject. In English an account of like attempts was given by Perronet Thompson, Cambridge, 1833, and is brought up to date in the charming volume, "Euclid and his Modern Rivals," by C. L. Dodgson, late Mathematical Lecturer of Christ Church, Oxford.

All this shows how ready the world was for the extraordinary flaming-forth of genius from different parts of the world which was at once to overturn, explain, and remake not only all this subject but as consequence all philosophy, all ken-lore. As was the case with the discovery of the Conservation of Energy, the independent irruptions of genius, whether in Russia, Hungary, Germany or even Canada gave everywhere the same results.

At first these results were not fully understood even by the brightest intellect. Thirty years after the publication of the book he mentions, we see the brilliant Clifford writing from Trinity College, Cambridge, April 2, 1870, "Several new ideas have come to me lately: First I have procured Lobatchewsky, 'E'tudes Géométriques sur la Theorie des Parallels' - - - a small tract of which Gauss, therein quoted, says : L' auteur a traité la matière en main de maitre et avec le véritable esprit geometrique. Je crois devoir appeler votre attention sur ce livre, dont la lecture ne pent manquer de vous causer le plus vif plaisir'. Then says Clifford: "It is quite simple, merely Enclid without the vicious assumption, but the way the things come out of one another is quite lovely."

The first axiom doubted is called a "vicious assumption," soon no man sees more clearly than Clifford that all are assumptions and none vicious. He had been reading the translation by Hoüel, published in 1866, of a little book of 61 pages published in 1840 in Berlin under the title Geometrische Untersuchungen zur Theorie der Parallellinien by a Russian, Nicolaus Ivanovitch Lobatchewsky, (1793-1856), the first public expression of whose discoveries, however, dates back to a discourse at Kasan on February 12, 1826.

Under this commonplace title who would have suspected the discovery of a new space in which to hold our universe and ourselves.

A new kind of universal space; the idea is a hard one. To name it, all the space in which we think the world and stars live and move and have their being was ceded to Euclid as his by right of pre-emption, description and occupancy; then the new space and its quick-following fellows could be called Non-Euclidean.

Gauss in a letter to Schumacher dated Nov. 28, 1846, mentions that as far back as 1892 he had started on this path to a new universe. Again he says: "La Géometrie non-Euclidienne ne renferme en elle rien de contradictoire, quoique, à première vue, beaucoup de ses resultats aient l'air de paradoxes. Ces contradictions apparents doivent etre regardées comme l'effet d'une illusion, due à l'habitude que nous avons prise de bonne heure de considérer la géométrie Euclidienne comme rigourous."

But here we see in the last word the same imperfection of view as in Clifford's letter. The perception has not yet come that though the non-Euclidean geometry is rigorous, Euclid is not one whit less so.

A clearer idea here had already come to the former roommate of Gauss at Göttingen, the Hungarian Wolfgang Bolyai. His principal work, published by subscription, has the following title:

Tentamen Juventutem studiosam in elementa Matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentique huic propria, introducendi. Tomus Primus, 1832; Secundus, 1833. 8o. Maros-Vàsàrhelyini.

In the first volume with special numbering, appeared the celebrated Appendix of his son Johann Bolyai with the following title:

Ap., scientiam spatii *absolute veram* exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) independentem. Auctore Johanne Bolyai de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Captaneo. Maros-Vàsàrhely., 1832. (26 pages of text).

This marvellous Appendix has been translated into French, Italian and German.

In the title of Wolfgang Bolyai's last work, the only one he composed in German, (88 pages of text, 1851,) occurs the following:

"Und da die Frage, *ob zwei von der dritten geschnittene Geraden wenn die Summa der inneren Winkel nicht=2R, sich schneiden oder nicht?*, niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhængige Geometrie abzusondern, und eine auf die Ja Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzen auf ein Wink auch in der ersten gültig seien."

The author mentions Lobatchewsky's Geometrische Untersuchungen, Berlin, 1840, and compares it with the work of his son Johann Bolyai, "an sujet duquel il dit: 'Quelques exemplaires de l'onvrage publié ici ont été envoyés à cette époque à Vienne, à Berlin, à Göttingen. . . De Goettingen le géant mathématique, [Gauss] qui du sommet des hauteurs embrasse du meme regard les astres et la profondeur des abimes, a écrit qu'il était ravi de voir exécuté le travail qu'il avait commencé pour le laisser après lui dans ses papiers.' "

Yet that which Bolyai and Gauss, a mathematician never surpassed in power, see that no man can ever do, our American Schuyler, in the density of his ignorance, thinks that he has easily done.

In fact this first of the Non-Euclidean geometries accepts all of Euclid's axioms but the last, which it flatly denies and replaces by its contradictory, that the sum of the angles made on the same side of a transversal by two lines may be less than a straight angle without the lines meeting. A perfectly consistent and elegant geometry then follows, in which the sum of the angles of a triangle is always less than a straight angle, and not every triangle has its vertices concyclic.

Gauss himself never published aught upon this fascinating subject, but when the most extraordinary pupil of his long teaching life came to read his inaugural dissertation before the Philosophical Faculty of the University of Göttingen, from the three themes submitted it was the choice of Gauss which fixed upon the one "Ueber die Hypothesen welche der Geometrie zu Grunde liegen." Gauss was then recognized as the most powerful mathematician in the world.

I wonder if he saw that here his pupil was already beyond him, when in his sixth sentence Riemann says, "therefore space is only a special case of a three-fold extensive magnitude," and continues: "From this, however, it follows of necessity, that the propositions of geometry cannot be deduced from general magnitude-ideas, but that those peculiarities through which space distinguishes itself from other thinkable three-fold extended magnitudes can only be gotten from experience. Hence arises the problem, to find the simplest facts from which the metrical relations of space are determinable—a problem which from the nature of the thing is not fully determinate; for there may be obtained several systems of simple facts which suffice to determine the metrics of space; that of Euclid as weightyest is for the present aim made fundamental. These facts are, as all facts, not necessary, but only of empirical certainty; they are hypotheses. Therefore one can investigate their probability, which, within the limits of observation, of course is very great and after this judge of the allowability of of their extension beyond the bounds of observation, as well on the side of the immeasurably great as on the side of the immeasurably small."

Riemann extends the idea of curvature to spaces of three and more dimensions. The curvature of the sphere is constant and positive, and on it figures can freely move without deformation. The curvature of the plane is constant and zero, and on it figures slide without stretching. The curvature of the two-dimentional space of Lobatchewsky and Bolyai completes the group, being constant and negative, and in it figures can move without stretching or squeezing. As thus corresponding to the sphere it is called the pseudo-sphere.

In the space in which we live, we suppose we can move without deformation. It would then, according to Riemann, be a special case of a space of constant curvature. We presume its curvature null. It would then lie between the sphere and pseudo-sphere. At once the supposed fact that our space does not interfere to squeeze us or stretch us when we move, is envisaged as a peculiar property of our space. But is it not absurd to speak of space as interfering with anything? If you think so, take a knife and a raw potato, and try to cut it into a seven-edged solid.

Farther on in this astonishing discourse comes the epoch-making idea, that though space be unbounded, it is not therefore infinitely great. Riemann says: "In the extension of the space-construction to the immeasurably great, the unbounded is to be distinguished from the infinite; the first pertains to the relations of extension, the latter to the size-relations.

That our space is an unbounded three-fold extensive manifoldness, is an hypothesis, which is applied in each apprehension of the outer world, according to which, in each moment, the domain of actual perception is filled out, and the possible places of a sought object constructed, and which in these applications is continually confirmed. The unboundedness of space possesses therefore a greater empirical centainty than any outer experience. From this however the Infinity in no way follows. Rather would space, if one presumes bodies independent of place, that is ascribes to it a constant curvature, necessarily be finite so soon as this curvature had even so small a positive value. One would, by extending the beginnings of the geodesics lieing in a surface-element, obtain an unbounded surface with constant positive curvature, therefore a surface which in a homaloidal three-fold extensive manifoldness would take the form of a sphere, and so is finite."

THE VOLUME OF THE PRISMOID AND THE CYLINDROID.

BY PROF. W. H. ECHOLS, ROLLA, MO.

In estimating the volume of earth work in the construction of Lines of Communication, a particular solid has occurred so frequently that engineers have given it a specific name; the Prismoid.

Whether the word was used to designate a definite geometrical solid prior to its adaption by engineers for that purpose, I have been unable to discover. The solid has been an extremely interesting one to engineers and much has been written by them upon the subject of its volume. No small subject connected with the profession has probably received so much labor and attention as this, in the direction of facilitating the computation of the volumes of these earthwork solids. The impracticability of an exact result so far as designing a mathematical surface which should coincide with the natural surface of the terrain was early recognized, and all efforts in dealing with the solid have been in the direction of offering approximate methods of computation, which yield results, the errors of which lie well within the limits which good practice demands.

In the sense of facilitating the computation of earthwork solids it is not the writer's intention to write in the present paper anything of a practical nature upon the subject chosen, but rather to discuss it from a purely geometrical point of view, believing, however, that such a discussion will contain matter which is not uninteresting to engineers, and which at the same time will be of practical benefit to them, insomuch as it will make more clear the advantages of the best methods now employed in practice for approximating to the volumes.

As to the practical feature of numerically computing the earthwork solids, that subject was shelved some fourteen years ago when Mr. Wellington and Prof. Davis published their works on Railway Earthwork*. It is because of a notice in the Engineering News of date, May 24, 1890, given to the second edition of the latter work, in which the Editor remarks "We are astonished that the author should be so ill-read in the literature of his subject as to state in his preface: '*The result of the prismoidal rule is for the first time obtained by a simple correction, without calculating the mid-section of these troublesome solids.*' By referring to p. 36 of Estimates of Railway Earthwork, by A. M. Wellington, published in 1874, he will find such corrections fully explained; and this was not the first.", that this paper was undertaken, the connection appearing in the sequel.

Both of these gentlemen base their methods of computation upon the same formula which is obtained by each in the same way. The final result reached is the method now employed in practice which in a few words may be expressed as follows: The mean area of the engineering prismoid is the average of its end areas, *corrected when necessary.* This correction is determined in each case, by computing the volume for *three-level* sections by the so-called prismoidal formula, then by the average of end-areas, the difference being the desired correction.

One in looking through engineering works cannot fail to be struck with the variety of definitions given to the prismoid solid, and in how few cases is the solid defined in a manner which fixes it in words which may be taken as a mathematical definition of the solid. As much as has been written about the prismoid in engineering journals in connection with the com-

*Computation from Diagrams of Railway Earthwork, A. M. Wellington. D. Appleton & Co., (1874.)

Formulae for Railway Earthwork, John W. Davis. New York Gilliss Brothers Pub., [1877.]

putation of earthwork volumes, no fixed mathematical definition of it has been agreed upon, and more or less confusion exists in the minds of engineers as to exactly what a prismoid is, beyond the definition given in the unabridged dictionaries where it is defined to be "a solid somewhat like a prism."

For the purposes of the present paper we shall use a definition for the prismoid which is derived from that given by Henck in his Fieldbook, Edition 1854. Where he says "A prismoid is a solid having two parallel faces, and composed of prisms, wedges and pyramids, whose common altitude is the perpendicular distance between the parallel faces." Let us adhere to this as defining the prismoid proper. More particularly expressed it appears as follows:

Definition:—A prismoid is a solid having two parallel plane polygons for bases, and whose side surface is made up of plane faces (triangles or quadrilaterals) formed by joining corresponding corners of the bases.

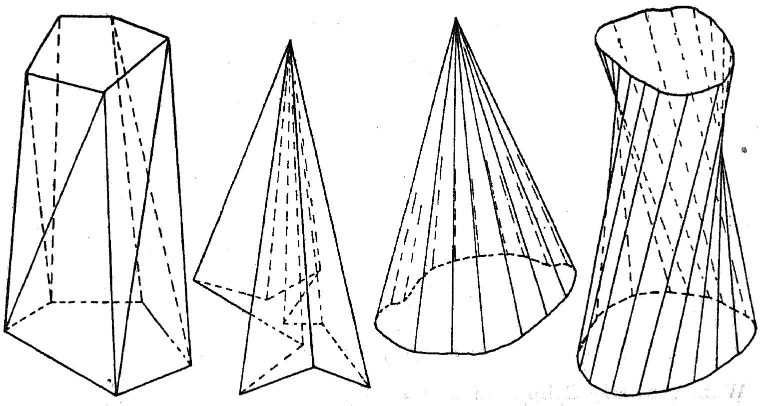
Using *corresponding* corners to denote any two corners, one of each base, such that the straight line joining them is an *edge* of the prismoid.

The property of the first definition follows immediately from the second; that is, it is evident that the solid just defined may be subdivided into prisms, wedges and pyramids; while the second definition serves to give a more definite idea of the shape of the solid as a geometrical figure and leads more directly to what follows below.

The cross-section or simply section of such a prismoid is the section by a plane parallel to the bases. The altitude or length of the prismoid is the perpendicnlar distance between the planes of the bases.

The Associate Pyramid:—If through any fixed point in the plane of one of the bases of the prismoid we draw straights parallel to the lateral edges of the prismoid to meet the plane of its other base in points which are taken to be the corners of a polygon in that plane, these lines define a pyramid whose vertex is in the plane of one base and whose base is in the plane of the other base of the prismoid. This pyramid is called the associate pyramid of the prismoid. It is easy to show that its volume is equivalent to the sum of the volumes of all the component pyramids of the prismoid.

In analogy, with the prism and cylinder of elementary geometry, if about the polygonal bases of the prismoid fixed plane closed curves be circumscribed, we have the following:



Definition:—The Cylindroid is the limit to which the Prismoid approaches when the number of the sides of the inscribed base polygons increases, and their magnitudes decrease, without limit.*

*Wiener in his Lehrbuch der Darstellenden Geometrie, Vol. II, page 471, defines a Cylindroid to be the scroll generated by a straight line guided by a director plane.

"Eine windschiefe Flæche mit einer einzigen, und zwar unendlich fernen Leitgeraden, also mit einer Leitebene, ist das Cylindroid."

In the Theory of Screws, English writers (Ball, Minchin, etc.) apply the name Cylindroid to a particular surface generated by two straights intersecting a third straight in a common point and normal to it, moving along

The plane closed curves become the bases of the cylindroid, and its side surface is a ruled surface. The associate pyramid of the prismoid thus becomes an associate cone to the cylindroid.

Under the above definitions we may now proceed to demonstrate the following theorem, which the writer first heard enunciated by Professor W. M. Thornton, of the University of Virginia, about ten years ago, but which he has never seen in print.

The mean area of the Prismoid (Cylindroid) exceeds the average area of its bases by one-sixth the area of the base of its associate pyramid (cone).**

Considering first the prismoid, we may give here first, the tentative method of identification employed usually to show

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it under fixed law. The angle between the first two straights varying periodically according to law. The equation to the furface is

$$z[x^2+y^2]-axy=0.$$

While the name Cylindroid has been thus differently appropriated to designate these higher mathematical surfaces, it has been thought to be no violation to use it in the present paper for the purpose of clearing up the railroad solid, it being very unlikely that any ambiquity will ever arise.

**In the Third Edition of one of the most recent text books on the Theory and Practice of Surveying we find in a foot note there the nearest approach to this theorem in print. In speaking of the different methods used for computing the earthwork volumes the foot note goes on to say:

"The method by 'mean end areas,' wherein the volume is assumed to be the mean of the end areas into the length, always gives too great a volume (except when a greater center height is found in connection with a less total width, which seldom occurs), the excess being one-sixth of the volume of the pyramids involved in the elementary forms of the prismoid."

This is wrong for the excess is one-half of the volumes of the pyramids involved in the elementary forms of the prismoid.

that Newton's Rule for Mean Area is applicable to the mean area of the prismoid.*

Let B_1, B_2, B_3 represent the area of the base of a component prism, wedge and pyramid of the prismoid respectively. The

volume of prism, $V_1=H[\frac{1}{2}(B_1+B_1)-\frac{1}{6}o]$,

wedge, $V_2=H[\frac{1}{2}(B_2+o)-\frac{1}{6}o]$,

pyramid, $V_3=H[\frac{1}{2}(B_3+o)-\frac{1}{6}B_3]$,

where H is the altitude. Using the same symbols, if B' and B'' are the areas of the bases of the prismoid and B_p that of its associate pyramid, then

$$B'+B''=\Sigma(B_1+B_2+B_3),$$

and
$$B_p=\Sigma B_3.$$

Therefore the volume of the prismoid is

$$V=H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_p].$$

Passing to the limit the volume of the cylindroid is therefore

$$V=H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_c].$$

Using B', B'', B_c to represent the areas of the bases of the cylindroid and its associate cone respectively.

The above is not a demonstration but merely an identification, and is only given here to parallel the process by which

*The so-called demonstration is as follows:

Using the same notations as above, let M be the area of the section mid-way between the bases, then for a component

prism $V_1=HB_1=\frac{1}{6}H[B_1+B_1+4M_1]$, $M_1=B_1$;

wedge $V_2=\frac{1}{2}HB_2=\frac{1}{6}H[B_2+o+4M_2]$, $M_2=\frac{1}{2}B_2$;

pyramid $V_3=\frac{1}{3}HB_3=\frac{1}{6}H[B_3+o+4M_3]$, $M_3=\frac{1}{4}B_3$.

Hence the volume of the prismoid is

$$V=\Sigma(V_1+V_2+V_3)$$
$$=\tfrac{1}{6}H(B'+B''+4M).$$

the result as shown in the foot-note is obtained, which is given by the text books as the prismoidal formula and the demonstration for it.

It would be more logical in giving an elementary deduction of a formula for the prismoid volume to proceed as follows, after first showing that the solid is composed of prisms, wedges and pyramids; thus the volume of a component prism is

$$V = HB_1 = \tfrac{1}{2}H2B_1,$$

wedge $$V = \tfrac{1}{2}HB_2,$$

pyramid $$V = \tfrac{1}{3}HB_3.$$

Hence the volume of the prismoid is

$$\begin{aligned} V &= H[\tfrac{1}{2}\Sigma(2B_1) + \tfrac{1}{2}\Sigma B_2 + \tfrac{1}{3}\Sigma B_3], \\ &= H[\tfrac{1}{2}\Sigma(2B_1 + B_2 + B_3) - \tfrac{1}{6}\Sigma B_3], \\ &= H[\tfrac{1}{2}(B' + B'') - \tfrac{1}{6}B_p], \end{aligned}$$ and so on.

It is only through the direct geometrical process for determining volumes of solids that we arrive in a satisfactory manner at the most appropriate formula for that purpose. Such a formula is then the true one for determining the volume of the particular solid in question as it is in general the simplest one.

Let us regard then the cylindroid as the highest type of the solids we have been considering, of which the prismoid and other degenerate forms are but particular cases. Thus we define the cylindroid independently of the prismoid as follows:

A cylindroid is the solid cut out from between two parallel planes by a moving straight, which finally returns to its initial position.

Alter idem;

A cylindroid is a solid whose bases are two parallel plane surfaces bounded by closed curves, and whose lateral surface is a regulus.

The regulus will in general be a scroll (warped).

Since a scroll is fixed when a linear director, one position of an element and the director cone is given. Then the cylindroid is fixed, given one base, one element and the associate cone, together with the position of the plane of the other base.

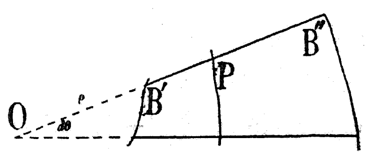
Again, if the cylindroid be given the associate cone is at once fixed, for from thc first definition it follows, that if through a fixed point in one of the planes a straight passes and moves so as to be always parallel to the straight which cuts out the cylindroid, the former cuts out the associate cone from the space between the two parallel planes.

To compute the volume cut from between two parallel planes by a moving straight, we proceed to find first the area of any cross-section (the area of a running section parallel to the base planes).

Project the moving straight and its traces with the planes on any plane parallel to the bases.

Let λ be the length of the projection of that part of the straight which is included between the parallel planes, its extremities being B′ and B″.

Let $d\vartheta$ be the angle through which the line λ turns in making a small shift, O the point of contact of λ with its envelope and ρ the distance of one end of λ from O.



Any plane parallel to the bases cuts the moving straight in a point which divides it in constant ratio, the projection (say P) of this point divides λ in the same ratio (say m/n,) this plane also divides the altitude H of the cylindroid in the same ratio.

Put
$$PB' = \frac{m}{m+n}\lambda = k'\lambda; \qquad PB'' = \frac{n}{m+n}\lambda = k''\lambda.$$

The area swept over by λ is the area included between the two curves traced by its extremities, so that if (B') and (B'') are the areas of the two closed curves traced, then $(B'')-(B')$ is the whole area swept over by λ. In like manner $(P)-(B')$ would be the whole area swept over by $k'\lambda$.

The area swept out by λ turning through $d\vartheta$ is

$$d(B'')-d(B')=\tfrac{1}{2}(\rho+\lambda)^2\,d\vartheta-\tfrac{1}{2}\rho^2\,d\vartheta$$

$$=\rho\lambda\,d\vartheta+\tfrac{1}{2}\lambda^2\,d\vartheta\,.$$

But $f(\lambda,\vartheta)=0$ is the polar equation to the base of the associate cone, hence

$$\tfrac{1}{2}\lambda^2\,d\vartheta=d(B_c)$$

and

$$d(B'')-d(B')=\rho\lambda\,d\vartheta+d(B_c) \qquad (1)$$

is the element of area included between the curves traced by B'' and B'.

In like manner

$$d(P)-d(B')=\tfrac{1}{2}(\rho+k'\lambda)^2\,d\vartheta-\tfrac{1}{2}\rho^2\,d\vartheta,$$

$$=\rho k'\lambda\,d\vartheta+\tfrac{1}{2}k'^2\lambda^2\,d\vartheta,$$

$$=k'\rho\lambda\,d\vartheta+k'^2d(B_c) \qquad (2.)$$

Multiplying (1) by k' and substracting (2) from the result, we have, observing that $k'+k''=1$,

$$d(P)=k'd(B'')+k''d(B')-k'k''d(B_c).$$

This is the relation which holds between the elementary areas of the curves traced by the points B', P and B'', referred to any system of coördinates.

If we integrate for a complete circuit of these points (closed curves) we have the relation between the bases of the cylindroid, the base of its associate cone and any cross-section of the cylindroid parallel to the planes of its bases.

Thus

$$P = k'B'' + k''B' - k'k''B_c.*$$

Put now $1 - k'$ for k'', then

$$P = B' + (B'' - B' - B_c)k' + B_c k'^2 \qquad (a).$$

Thus P is a quadratic function of k'. If now k' be allowed to vary continuously from 0 to 1, then P becomes a running cross-section taking in succession all the values of the sections from one base to the other. The average of all of these is then the mean area of the cylindroid. Thus the mean area of the cylindroid is in symbols

$$\Omega = \frac{1}{1-0}\int_0^1 P\,dk',$$

$$= \int_0^1 [B' + (B'' - B' - B_c)k' + B_c k'^2]\,dk',$$

$$= \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c.$$

Otherwise by the the ordinary geometrical process, let h be the distance of the cross-section P from the plane of one of the bases (say B'), then $h/H = k'$, substituting in (a)

$$P = B' + \frac{B'' - B' - B_c}{H}h + \frac{B_c}{H^2}h^2.$$

The section of a cylindroid is therefore a quadratic function of its length. The volume of the solid is then

$$V = \int_0^H P\,dh.$$

Putting in the second member above for P and operating we have as before

$$V = H[\tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c].$$

This then is the rational formula for computing the volume of any cylindroid or prismoid. It should therefore be expected

*This is Holditch's Theorem.

to give the volume of the solid, with less labor than is required by any more comprehensive formula.

It is easy to eliminate B_c between equations

$$P = k'B'' + k''B' - k'k''B_c$$

and

$$\Omega = \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c,$$

thus getting the mean area Ω in terms of the base areas and that of any cross-section, as for example putting $k' = k'' = \frac{1}{2}$, then P becomes M the mid-section. Hence

$$M = \tfrac{1}{2}(B'' + B') - \tfrac{1}{4}B_c,$$

$$\Omega = \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c.$$

From which by subtraction we see that the mean area differs from the mid-area by one-twelfth the base of the associate cone.

Eliminating B_c we have

$$\Omega = \tfrac{1}{6}(B' + B'' + 4M).$$

This is the form of the so-called prismoidal rule, more generally known as Simpson's Rule, but which is really due to Newton [*Methodus Differentialis*]. It may be found deduced in any good work on Integral Calculus [Todhunter, p. 158]. It is mis-nomer to call it the prismoid formula, for it applies not only to the cylindroid and all of its degenerate forms but applies as well to a large class of solids of a higher order. One would be as well justified in calling the formula above deduced for the cylindroid the "conical formula" because it happens to give the volume of a cone-frustum, as calling Newton's Rule for mean area the "prismoid formula" because it gives the mean area of the prismoid which is only one of the degenerate forms to which the rule applies.

We have seen above that it is a characteristic property of the cylindroid (prismoid), that its cross-sectional area is a quadratic function of its length, therefore the formula for the mean area of the cylindroid is also the formula which gives the mean area of any solid whose cross-sectional area is a linear function of its length, but the converse is not true, for in such solids as the latter (wedges and conoids) there is no associate cone.

Newton's Rule for mean area gives not only the mean area for solids whose sectional areas are linear and also quadratic functions of their lengths, but also of all solids whose sections are *cubic* functions of their lengths. It is therefore far more comprehensive than the true prismoid formula. The demonstration of this is also of the nature of an identification; it is as follows: (Todhunter Int. Cal., p. 173).

Let there be a solid such that the area of a section made by a plane parallel to a fixed plane and at a distance l from it is always

$$\mathrm{P}=a+bl+cl^2+dl^3, \qquad (1)$$

where a, b, c and d are constants.

Let three equidistant sections of the solid B′, M. B″ be made by the fixed plane and two others parallel to it in order. Then the volume of the portion of the solid included between the two extreme sections is

$$\mathrm{V}=\int_0^{\mathrm{L}} \mathrm{P}\,dl,$$

$$=a\mathrm{L}+\tfrac{1}{2}b\mathrm{L}^2+\tfrac{1}{3}c\mathrm{L}^3+\tfrac{1}{4}d\mathrm{L}^4.$$

Where L is the length of the solid, *i. e.*, perpendicular distance between the planes of B′ and B″. The mean area is therefore

$$\Omega=\mathrm{V}/\mathrm{L}=a+\tfrac{1}{2}b\mathrm{L}+\tfrac{1}{3}c\mathrm{L}^2+\tfrac{1}{4}d\mathrm{L}^3. \qquad (2).$$

But by (1)

$$\text{if } l=0; \qquad \mathrm{P}=\mathrm{B}'=a.$$

$$\text{if } l=\tfrac{1}{2}\mathrm{L}; \qquad \mathrm{P}=\mathrm{M}=a+\tfrac{1}{2}b\mathrm{L}+\tfrac{1}{4}c\mathrm{L}^2+\tfrac{1}{8}d\mathrm{L}^3.$$

$$\text{if } l=\mathrm{L}; \qquad \mathrm{P}=\mathrm{B}''=a+b\mathrm{L}+c\mathrm{L}^2+d\mathrm{L}^3.$$

Therefore

$$B'+4M+B''=6a+3bL+2cL^2+\tfrac{3}{2}dL^3$$

and comparing with (2) we find

$$\Omega=\tfrac{1}{6}(B'+B''+4M).$$

The formula involving the base of the associate cone being of a less general character than that just written it is reasonable to expect of it more simplicity in application, as the sequel shows.

Consider for a moment a prismoid. Project it on a plane parallel to the bases. In this plane refer all points to any system of rectangular axes. Then if x', y' and x'', y'' be coördinates of a pair of corresponding corners in the bases. The coördinates of the corresponding corner in the mid-section will be

$$\tfrac{1}{2}(x'+x''); \qquad \tfrac{1}{2}(y'+y''),$$

while the coördinates of the corresponding corner of the base of the associate pyramid are

$$x'-x''; \qquad y'-y''.$$

The computation of the areas of the bases is the same in either case, while in order to compare the labor of computing the area of the mid-section with that required for the base of the pyramid, it is only necessary to see that in the respective coordinates we deal with sums in the one case and differences in the other, with the additional practical advantage always present that in the latter case the formula for mean area is in the shape of a correction applied to the average of end areas, the base of the pyramid in practical cases being small, whereas the mid-area generally exceeds the average of end areas.

While it is of no practical importance to the engineer it may nevertheless be interesting to apply the foregoing for the sake of illustration to the particular case of the railway prismoid.

The engineer in surveying his cross-section, refers the polygon to a system of rectangular axes in its plane and measures the coördinates of its corners with respect to the road bed line and the vertical through its center as axes of abscissa and ordinate respectively, calling them in order *distance-out* and *height*. Furthermore he takes no cognizance of the algebraic change of sign in coördinates, but merely calls them rights and lefts, cuts and fills respectively. The uniform method adapted for recording the field notes preserves the identily of the section.

Thus if b be half the road bed, h and m the height and distance-out to the right of center, k and n the corresponding measurements to the left, the record of the cross-section appears complete in the adapted form.

$$\frac{0}{b}\ \frac{k_s}{n_s} \cdots \frac{k_1}{n_1}\ \frac{d}{0}\ \frac{h_1}{m_1} \cdots \frac{h_s}{m_s}\ \frac{0}{b}$$

where d is the center height.

The area of any polygon in terms of the coördinates of its n corners being

$$2A = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + - - - + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix},$$

it is easy to see that the engineer's record of his cross-section is really a determinant for its double area. Thus the double area of the cross-section above recorded is

$$\frac{0}{b} \times \frac{k_s}{n_s} \times - - - \times \frac{k_1}{n_1} \times \frac{d}{0} \times \frac{h_1}{m_1} \times - - - \times \frac{h_s}{m_s} \times \frac{0}{b}.$$

In which the heavy lines join factors of positive product, dotted lines those of negative product. The sum of all the products is double the area.*

*This formula for the area of an irregular section was first given in Engineering News, Vol. XX, No. 39, in an article under the heading "A Cross-Section Mnemonic."

The cross-sections being written successively in the note book, each pair represents the bases of a prismoid whose lateral edges are noted, while on the ground by the engineer, by joining the coördinates of corresponding corners by a line in the notes.

Let the bases of a prismoid be

$$\begin{array}{cccccc}
\frac{0}{b} & \frac{K_s}{N_s} & & \frac{D}{/0} & \frac{H_1}{/M_1} \ \frac{H_s}{/M_s} & \frac{0}{b} \\[2ex]
\frac{0}{b} & \frac{k_s}{n_s} & \frac{k_1}{n_1} & \frac{d}{0} & \frac{h_1}{m_1} & \frac{0}{b}.
\end{array}$$

Coördinates in the same vertical are presumed to correspond without further indication.

The double mid-area for such a prismoid is to be computed from

$$\frac{0}{b}\ \frac{\frac{1}{2}(K_s+k_s)}{\frac{1}{2}(N_s+n_s)}\ \frac{\frac{1}{2}(K_s+k_1)}{\frac{1}{2}(N_s+n_1)}\ \frac{\frac{1}{2}(D+k_1)}{\frac{1}{2}n_1}\ \frac{\frac{1}{2}(D+d)}{0}\ \frac{\frac{1}{2}(H_1+d)}{\frac{1}{2}M_1}\ \frac{\frac{1}{2}(H_1+h_1)}{\frac{1}{2}(M_1+m_1)}\ \frac{\frac{1}{2}(H_s+h_1)}{\frac{1}{2}(M_s+m_1)}\ \frac{0}{b},$$

while that of the base of the associate pyramid comes from

$$\frac{K_s-k_s}{N_s-n_s}\ \frac{K_s-k_1}{N_s-n_1}\ \frac{D-k_1}{-n_1}\ \frac{D-d}{0}\ \frac{H_1-d}{M}\ \frac{H_1-h_1}{M_1-m_1}\ \frac{H_s-h_1}{M_s-m_1}.$$

Employing the same rule in either case as that given for the area of any ordinary cross-section, noticing that in the latter case the subtractions may change the sign of some of the products.

To apply the results to a numerical case, take the example in Henck's Field book, Art. 122, which he uses to compare methods.

$$B''=\frac{0}{9}\ \frac{4}{15}\ \frac{8}{/0}\ \frac{12}{/27}\ \frac{0}{9}=240$$

$$B'=\frac{0}{9}\ \frac{8}{21}\ \frac{13.6}{0}\ \frac{10}{24}\ \frac{0}{9}=387$$

$$M=\frac{0}{9}\;\frac{6}{18}\;\frac{8}{10.5}\;\frac{10.8}{0}\;\frac{12.8}{13.5}\;\frac{11}{25.5}\;\frac{0}{9}=355.55$$

$$B_p=\frac{4}{6}\;\frac{0}{21}\;\frac{5.6}{0}\;\frac{1.6}{-27}\;\frac{-2}{-3}=-88.2$$

Thus
$$\Omega=\tfrac{1}{6}(B'+B''+4M)=\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_p$$
$$=328.2\,.$$

The advantage in the computation lies so largely in favor of B_p as against M, that neglecting the advantage to be derived from the former as a correction to $\frac{1}{2}(B'+B'')$ it is preferable to use the simple method in cases where the actual volumes are to be computed.

Regularly and ordinarily in practice only three-level sections occur, and even then the computation of volume is further simplified by conceiving the surface ground to be determined by gauche quadrilaterals through each of which is passed a hyperbolic paraboloid, thus for each such quadrilateral we have one less corner in the mid-area and also in the base of the associate cone than would have occurred had the Henck prismoid been used instead. Evidently the introduction of the hyperbolic paraboloids does not interfere with the mid-area and the base of the associate pyramid remaining polygons, for in this surface one set of generators is parallel to the bases of the solid the other set for each surface moves always parallel to a fixed plane, therefore the corresponding element of the associate pyramid moves in a plane and traces a straight in the plane of the base. The reason for this simplification is not merely to save labor, but because in fact the volume for any gauche quadrilateral as determined by its hyperbolic paraboloid is exactly the arithmetical mean of the volumes which are determined by considering the diagonals of the quadrilateral successively as edges of a Henck prismoid. To prove this it is only necessary to prove the following:

Theorem:—The volume of the solid bounded by plane triangles, whose edges are the sides and diagonals of a gauche quadrilateral, is bisected by the hyperbolic paraboloid passing through the sides of the quadrilateral.

Let a and c be two opposite sides of the quadrilateral, the solid is composed of two pairs of triangles, a non-contiguous pair meeting in a, the other pair in c. Any plane parallel to a and c cuts the solid in a parallelogram, for it cuts the planes which meet in a in straights parallel to a, and those which meet in c in straights parallel to c. This parallelogram moving parallel to itself generates the solid. But the plane of this parallelogram always contains the element of the hyperbolic paraboloid of the (a, c) generation, and since this meets the other two sides of the quadrilateral it must be the diagonal of the parallelogram, dividing that figure into two equal triangles, each of which generates half of the solid.

Thus the ignoration of the diagonals, while in one particular prismoid may give an approximation to the earth volume not within the limit of error, in any series of consecutive prismoids must give a volume very near the true earth volume, since the cross-ridge and valley lines are just as likely to occur as one diagonal as the other.

It is under these assumptions then that engineers compute earth work volumes. Sections being reduced to three-level sections, the computation of mid-area and base of pyramid are correspondingly simplified. Thus in the regulation prismoid

$$\frac{\mathrm{o}}{b}\ \frac{\mathrm{K}}{\mathrm{N}}\ \frac{\mathrm{D}}{\mathrm{o}}\ \frac{\mathrm{H}}{\mathrm{M}}\ \frac{\mathrm{o}}{b}$$

$$\frac{\mathrm{o}}{b}\ \frac{k}{n}\ \frac{d}{\mathrm{o}}\ \frac{h}{m}\ \frac{\mathrm{o}}{b}.$$

The mid-section becomes

$$\frac{\mathrm{o}}{b}\quad \frac{\frac{1}{2}(\mathrm{K}+k)}{\frac{1}{2}(\mathrm{N}+n)}\quad \frac{\frac{1}{2}(\mathrm{D}+d)}{\mathrm{o}}\quad \frac{\frac{1}{2}(\mathrm{H}+h)}{\frac{1}{2}(\mathrm{M}+m)}\quad \frac{\mathrm{o}}{b}.$$

The area of which can be computed as above, or may now be written out as a rule, thus

$$8M = 2b(\Sigma S) + (D+d)\,\Sigma O.$$

Eight times mid-area is $2b$ times the sum of side-heights $+(D+d)$ times the sum of the distance-out.

The consideration of the mid-area is useless and unnecessary for the base of the pyramid is

$$\frac{K-k}{N-n}\;\frac{D-d}{0}\;\frac{H-h}{M-m},$$

and its double area in algebraic form is

$$(D-d)\,(M+N-m-n)\,.$$

One-twelfth of this expression (which may be negative) subtracted from the average of end-areas gives the true mean area. It is upon this basis that the tables referred to have been computed. A formula may be written down at once for the correction to the average end-areas for any given cross-sections, but it would in general be too complicated for use.

An interesting point in connection with the cylindroid (prismoid) is the distance of its center of gravity from the plane of the mid-section, a value which is used in explaining the question of *long haul*. The formula for the running cross-section lends itself to an easy deduction of this.

Thus if X be the distance of the center of gravity from the plane of the base from which h is measured we have by the ordinary formula,

$$VX = \int_0^H Ph\,dh,$$

putting in the value of P in terms of h from above, and for V its value $H[\frac{1}{2}(B'+B'')-\frac{1}{6}B_c]$ we have, after integration,

$$H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_c]\,X = H^2\,[\tfrac{1}{6}B'+\tfrac{1}{3}B''-\tfrac{1}{12}B_c].$$

The distance of the center of gravity from the mid-section is

$$x = X - \tfrac{1}{2}H.$$

Substituting in the above we get

$$x = \frac{H}{6} \frac{B'' - B'}{B'' + B' - \frac{1}{3}B_c}.$$

Whence the approximate formula used by the engineer

$$x = \frac{H}{6} \frac{B'' - B'}{B'' + B'}.$$

Since $\frac{1}{3}B_c$ is practically small when compared with $B'' + B'$.

THE BEGINNING OF MATHEMATICS.

By Prof. W. B. RICHARDS, Rolla, Mo.

II GEOMETRY.

—μὴ εἶναι βασιλικὴν ἄτραπον ἐπὶ γεωμετρίαν.

Euclid. ap. Procl. Diadoch.

It is unfortunate that the adjective—Greek—in prevailing use to designate whatever pertains to the people of whose intellectual labors we are about to speak, should suggest but a part—and what at some periods was a minor part—of the territory through which this people was scattered, and in which their mental activity had its first field. The Hellenes themselves knew their country, as their descendants still know it, only as Hellas. The name Greece, given to it by the Romans and adopted by common consent of later times, is derived from *Γραικοί*, the name of a small tribe with whom the Romans first came in contact. It is not mentioned by any writer earlier than Aristotle. The employment of the term Greek, while sanctioned by usage, is apt to be misleading to one better acquainted with modern geography than with ancient history. The national spirit was in a largely measure wanting among the Hellenes, and the ties which they recognized were based upon ethnic, rather than geographic relations. The name Hellas was broad enough to include not merely the dwellers in the little peninsula which now bears the name of

Greece, but all, under whatever sky, who might claim descent from them—not merely the walker in the groves at Athens, the hardy Spartan, or the Messenian mountaineer, but him as well of the same stock who had his household gods by Leucadian steep, or breathed the soft air of Italian Sybaris. Indeed the first fruitage of what was to be so glorious a springtime sprang upon other soil than that which lies between Olympus and Taenarum. It was among the colonies, not on the main-land, that were made the first steps in Literature, in Philosophy and in Science. Except the Boeotians, Hesiod and Pindar, no poet of the first rank (if the former's theogony and rough and ready expressions of practical wisdom entitle him to be named as an exception) acknowledged as his home what we know as Greece until the time of Aeschylus. Homer, if we admit that there was one such man who wrote the poems attributed to him, was an Ionic Greek, living in Asia Minor or on one of the islands that fringe its coast. The biting iambics of Archilochus, the noble lyrics of Simonides, Anacreon's praise of Love and Wine, the mutual sighs of Alcaeus and soft, slandered Sappho, all come from lips that learned to lisp numbers amid the Aegean isles. All the early philosophers from Thales to Sacrates, that is for two centuries, were natives of the one or the other of the Hellenic colonies.

The Ionic Greeks, who had their seats along the middle part of the western coast of Asia Minor and in the islands adjácent to it—"Sons of Javan", as the Scripture calls them—were the earliest movers in the work of Greek culture. Their characteristics as a people and their situation combined to give them this precedence. The Ionians were distinguished among the other Greeks for their quickness, their viracity, their readiness to receive impressions. The stuff of which they were made was far more fictile than that of their Aeolic or Doric kinsfolk. They possessed in the highest degree among Greeks the qualities that distinguished the Greeks from their contemporaries.

Their location, in the direct track of the western process of civilization, and their commercial relations with the Egyptians and the Phoenicians, contributed to make them first among Aryans to feel the impetus toward scientific investigation which an acquaintance with the attainments of these people would give.

The meagreness of the contemporary historical records does not enable us to speak with definiteness and certainty as to the exact connection between the incipient Greek culture and the achievements of its predecessors. Ueberweg (Hist. of Philosophy, Vol. 1., p. 31,) says: "To what extent the philosophy of this age (and hence the genesis of Greek philosophy in general) was affected by Oriental influences, is a problem whose definite solution can only be anticipated as the result of the further progress of Oriental and, especially, of Egyptological investigations. It is certain, however, that the Greeks did not meet with fully developed and completed philosophical systems among the Orientals," The same general fact is true of Science. Nor are the traditions of either the Greeks or the Orientals entirely trustworthy. It would not be strange if the early Greeks, anxious to lend a flavour of antiquity to their teachings, should have attributed their origin to the Egyptians, nor if the national pride of this latter people first consented to the attribution, and then insisted upon it, until they, and the world at large, placed far more stress upon the indebtedness of the younger to the older people than is justified by the facts. The work of the Orientals is not to be neglected in estimating the influences that brought about the beginnings of Science, yet on the other hand we need to guard against the danger of ascribing to it a part in the history of Science in general, and Mathematics in particular, beyond that which it really played. What they did was a leading up to Science rather than a beginning of it, and the debt due to them from the Greeks and all later nations was not comparably so much for actual contribution as it was for suggestion and incentive.

The beginning of Science is signalized by the appearance for the first time of a single name in connection with the advancement of knowledge. The Assyrians had an Astronomy, with copious records of observations made, but no astronomer; the Egyptians had an inkling of Geometry, but no geometer. Some progress in learning may be made under the push of natural laws by a people, working without concert, yet happening the one occasionally to cap the discovery of another with a greater; but no body of thought assumes the proportions of a science until its scattered fragments have been collected and fused together in the crucible of a single brain.

The same venerable personage stands at the head of the long list of philosophers, astronomers and mathematicians. Indeed at this early period to be one of these was well nigh being the others.

Thales of Miletus was born in the Ionic city of that name on the western coast of Asia Minor about the year 640 B. C. Herodotus (Book I., c. 170) says he was of Phoenician descent. Diogenes Laertius gives Plato as authority for the tradition that his ancestry might be traced back to Cadmus, who first introduced letters into Greece, and Zeller agrees with this view. Another account makes him out a native Milesian of pure Greek blood. Whether or not his family relations were such as would involve a connection with the people by whom his compatriots were being imbued with learning, the circumstances of his birth placed him in the immediate path of the westward flowing stream of knowledge. Thales, who enjoyed among the Greeks a reputa-

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*Lives of the Philosophers, I., 1.

tion for wisdom which we know not whether to attribute more to his own worth or to the obscurity of the period, was by common consent acknowledged first of the seven sages. His most valid claim to the admiration of his contemporaries and posterity as well, was the alleged prediction of an eclipse of the sun, which took place in 585 B. C. This was the same obscuration otherwise famous in history as having occurred just as the Lydian and Median Kings were about to join battle on the banks of the river Halys, and which so filled these barbarous potentates with awe that they at once declared a mutual peace. Herodotus, who was too fond of a good story to be embarrassed by considerations as to its truth, recites this legend (B'k. I, c. 74), and later authorities follow him. There are, however, we regret to say, serious, if not insurmountable, difficulties in the way of our lending credence to this pleasant narrative. The state of Astronomical science at the time would scarcely warrant the belief in the possibility of so exact a calculation. It is not at all unlikely that Thales was acquainted with the Assyrian "Saros," a period of eclipses covering 233 lunations, or even the longer period of 600 years. But while eclipses of the moon were predicted by means of these cycles it is disputed whether eclipses of the sun could be foretold in the same way. To have predicted this phenomenon for any definite locality moreover, would have involved a knowledge of the sphericity of the earth, which was sufficiently foreign to Thales' conceptions. Plutarch says that Thales was engaged in commerce, and all the authorities agree that in this way he was led to Egypt, and became acquainted with the Egyptian rudiments of Geometry. Diogenes Laertius quotes Hieronymus of Rhodes as asserting that he never had any teacher save when he went to Egypt and associated with the priests.*

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*Lives of Philosophers, I., 6.

Hence he obtained the inspiration of his Geometric investigations; his speculative spirit seized upon the crude fragments of truth tortured from centuries of experience and observation, and began to build beyond them into the Empyrean. Proclus (Ad. Euclid. p. 19) says:

"Θαλῆς δὲ πρῶτον εἰς Αἴγυπτον ἐλθὼν μετήγαγεν εἰς τὴν Ἑλλάδα τὴν θεωρίαν ταύτην καὶ πολλὰ μὲν αὐτὸς εὗρε, πολλῶν δὲ τὰς ἀρχὰς τοῖς μετ' αὐτὸν ὑφηγήσατο, τοῖς μεν καθολικώτερον ἐπιβάλλων, τοῖς δὲ αἰσθητικώτερον."*

The following propositions are attributed to Thales:

(1.) The circle is halved by its diameter.

(2.) The angles at the base of an isosceles triangle are equal.

(3.) The vertical angles formed by the intersection of two right lines are equal.

(4.) Two triangles are equal when they have one side and two angles of the one equal respectively to the corresponding parts of another.

(5.) The angle inscribed in a semi-circle is a right angle.

Diogenes Laertius says, "Pamphile relates that he (Thales), having learnt Geometry from the Egyptians, was the first person to describe a right angled triangle in a circle, and that he sacrificed an ox in honor of his discovery. But others, among whom is Apollodorus, the calculator, say that it was Pythagoras who made this discovery. It was Thales also who carried to their greatest point of advancement the discoveries which Callimachus in his iambics says were first made by Euphebus the Phrygian, such as those of the scalene angle and of the triangle, and of other things which relate to investigations about lines."**

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*Thales, after having journeyed into Egypt, brought back this science (Geometry) to Greece and both discovered many things himself and handed down to his successors the elements of many things, approaching some in a more general manner, some in a more experimental.

**Lives of Philosophers, I., 25.

(6.) The homologons sides of similar triangles are in proportion. Plutarch distinctly ascribes this to him.

We are confronted with the difficulty of which we have spoken in attempting to determine what part of the enunciations accredited to Thales was derived from his intercourse with the Egyptian priests, and what was original with him. We can readily see how the conclusions, (1), (2) and (3), could be reached inductively from observation of particular cases, and might belong to that portion of his teachings at which Proclus says he arrived *αἰσθητικώτερον*—in a more sensible (empiric) manner—a portion which may fairly be assumed to stand for his immediate acquisition from the Egyptians; while (4), (5) and (6) would seem to belong to that part proved *καθολικώτερον*—more generally—and to be the product of his own invention.

The proof of (5) [Euclid I, 31,] involves (2) and the principle that the sum of the angles of any triangle is equal to two right angles. This would demand that Thales should be acquainted with the last named proposition—that is if a general proof of (5) was offered. Proclus asserts that the theorem concerning the angles of a triangle's being equal to two right angles was first proved in a general way by the Pythagoreans, but it was probably known to early mathematicians as a fact of observation.

Two applications of this new instrument, Geometry just being fitted to the worker's hand, to the solution of practical problems—marvelous enough they must have seemed to the ancients—are handed down as having been made by Thales. These were the determination

(1) of the distance of a ship at sea;

(2) of the height of the pyramids by their shadows.

These problems are interesting, besides in other respects, as showing the influence of environment in determining the direction of mental effort, and confirming the principle, upon which we touched in the preceding paper, that inventions in the field of science spring from the suggestion of practical questions. The residence of Thales upon the coast and among a maritime people, naturally presented the first problem to his inquiring mind, and furnished an incentive to the solution of it which would have been absent had he spent his days inland; while his travels in Egypt, beneath the shadow of the lofty pyramids, could not fail to stir his spirit up to an attempt to compass what seemed the impossible feat of measuring those inaccessible heights. And it is more natural to suppose that the important general theorem that the sides of equiangular triangles are proportional, which it is generally assumed that the solution of these problems presupposed, was discovered in the attempt to solve them, than that it occurred to Thales in a purely abstract way, and that the questions were afterwards resolved by its aid.

Diogenes Laertius quotes Hieronymus of Rhodes as saying that "He measured the pyramids, watching their shadow and calculating when they were of the same size as that was." Others give an account of the feat which would involve the use of Theorem (6) alone. Obviously enough both of these problems might be solved without using (6), by means of (2).

So much stands accepted in history as the tangible work of Thales. But remarkable as were these achievements in comparison with aught that had been done before, they in themselves mark but a fraction of the service of Thales to later science. The prime element of this contribution was the institution of a method of inquiry which passed beyond the individual instance, and sought to lay an unchanging foundation in the principles of abstract truth. Science was set on foot and might run its victorious course.

The immediate successors of Thales in the Ionic school of Philosophy, while they took up the physical speculation in which he had set an example, do not seem, so far as any records show, to have made any advancement in Mathematical invention. The next step in the development of Geometry was the work of the Italic school—to accept the ancient geographical division—that is, of Pythagoras and his followers. The name of Pythagoras is one suggestive at once of myth and mystery. The real figure of the man has come down through history girt about with an increasing nimbus of mystic tradition. He is the hero of a cycle of philosophic legend. Not only his own age but later ages have ascribed to him the possession of supernatural power, and have painted him and his school in the colors of romance. Out of the mass of tradition which has been handed down to us, it is impossible to assert confidently how much is fact and how much fiction. His life, if the reports of the ancient historians are accepted, was as varied as the range of his speculations.

He was the son of Mnesarchus, the Tyrian, a seal-engraver, and was born in the island of Samos about 582 B. C. He studied at Lesbons under Pherecydes and at Miletus under Thales and Anaximander. He visited Egypt and spent twenty-seven years at Memphis and Thebes communing with the priests and sages. When Cambyses over-ran Egypt in 525 B. C., Pythagoras was among the captives, and was carried off to Babylon, where he was held as a slave for some years. Here he became acquainted with the learning of the Chaldaeans, and gained as well an introduction to the religion of the Hindus. Having obtained his liberty, he visited in turn Crete, Sparta, Elis, and Delphi, and returned to his native isle to establish a school. Discontented with the tyranny of Polycrates, he proceeded to Italy and founded at Crotona "in the house of Milo", a school which soon attracted a large number of attendants.* Interference in local politics caused his own

*Diogenes Laertius says three hundred.

banishment, the burning of his house and the dispersion of his followers. He retired to Metapontum where he died soon after, (about 490 B. C.)

Such the narrative that might be constructed by one liberally credent. The statements as to his travels and adventures in the East, it has been observed, may either be statements of fact or an invention to connect his teachings with earlier originals. There are reasons to believe that he did travel in Egypt; the rest is more doubtful. Cicero says, (De. Fin., V. 29, 87) "*Aegyptum lustravit.*" He also says that Pythagoras came to Crotona in Ol. 62. 4 (529 B. C.,) (*Rep.* II. 15). If this be true we should have to give up the story of his Babylonish captivity. We must ever regret that the history of Pythagoras' life, written by Theano, a beautiful young girl whom he espoused in his seventieth year, has not come down to us.

The society which Pythagoras founded was on its scientific side a school of philosophic inquiry and instruction, while on its ethic side it partook of the nature of a religious brotherhood. The search for truth was combined with a rigid personal discipline. Intending disciples were said to have been subjected to a long period of probation, of which strict obedience and absolute silence were the cardinal features. Diogenes Laertius says that this period was five years. The use of animal food was permitted only within certain restrictions. Certain vegetables were tabooed, and celibacy was inculcated. Thus we see that the ancient Pythagoras and the modern Tolstoi are alike as regards both theory and practice.

The speculations of the school took a wide range—over philosophy, astronomy, mathematics, music. Best known, perhaps, of their tenets is that of the transmigration of souls, which their great founder is thought to have imbibed in his Oriental wanderings. He, himself, claimed to be a son of Mercury, and to have existed in many previous shapes. He said that Mercury offered him any gift save immortality and that accordingly he requested that, whether living or dead, he might preserve the memory of what had happened to him. So was his existence continuous.*

The philosophical conceptions of the Pythagoreans were strangely blended with arithmetic considerations. The whole system of the universe was held to depend upon the relations between numbers. "The Pythagoreans seem" says Aristotle, "to have looked upon number as the principle and, so to speak, the matter of which existences consist." "Number," says Philolaus, (the successor of Pythagoras), "is great and perfect and omnipotent, and the principle and guide of divine and human life."†

Proclus states, in his commentary on Euclid's elements, that the word "mathematics" originated with the Pythagoreans. The same author says that the Pythagoreans made a four-fold division of mathematical science, its parts corresponding to Arithmetic, Music, Geometry and Astronomy, respectively.‡ Diogenes Laertius relates that "It was Pythagoras also who carried Geometry to perfection, after Moeris had first found out the principles of the elements of that science as Aristiclides tells us in the second book of his History of Alexander."§ The Pythagoreans defined a point as "Unity having position." They showed that the plane around a point is completely filled by six equilateral triangles, four squares, or three regular hexagons. (Proclus). Eudemus attributes to them the theorem that the interior angles of a triangle are equal to two right angles, and gives their proof, which is substantially the same as that given by Euclid. We have it stated upon the same authority (quoted by Proclus in his commentary) that the problems relating to the application of areas, the construction of

*Diogenes Laertius. Lives of Philosophers, B'k. VIII, c. 4.

†Encyclopædia Britannica, Vol. XX. p. 144.

‡Encyclopædia Britannica, Art. Pythagoras, p. 146.

§Diogenes Laertius, Lives of Philosophers, VIII, 11.

the five regular solids, and the discovery of irrational quantities were all due to Pythagoras. Three of the five regular solids, the tetrahedron, the cube and the octahedron, were known to the Egyptians, and occur in their architecture. Pythagoras discovered the other two, the dodecahedron and icosahedron, and shewed how to construct them all. The discovery of the existence of irrational quantities was one of the most notable made by the Pythagorean school; it may have arisen from an attempt to express the length of the diagonal of a square in terms of a side. It paved the way for the general treatment of proportion found in Euclid—a treatment which holds as well for incommensurable as for commensurable magnitudes.

The theorem best known in connection with the name of Pythagoras—in fact frequently cited as the Pythagorean theorem—is that which asserts that "the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides." This proposition is distinctly attributed to Pythagoras by Vitruvius, Diogenes Laertius, Proclus and Plutarch. Diogenes Laertius says, "And Apollodorus, the logician, records of him that he sacrificed a hecatomb when he had discovered that the square on the hypotenuse of a right-angled triangle is equal to the squares of the sides containing the right angle. And there is an epigram which is couched in the following terms:

> "When the great Samian sage his noble problem found,
> A hundred oxen dyed with their life blood the ground."

This, it will be observed, is just one hundred times the amount of gore which the same author represents Thales as having spilled after having inscribed a right angle in a circle. Plutarch in his work on "Isis and Osiris," dealing with the mysteries of Egyptian religion and learning, asserts that the ancient Egyptians knew that a triangle whose sides contain three, four and five parts respectively is right-angled, and that the square

*Lives of Philosophers, VIII, 11. (Yonge's Translation.)

on the longer side is equal to the sum of the squares on the other two. It has been very plausibly suggested, too, that the fact might have become known to them in a special case by the observation of the tesselated floors common in Egyptian buildings. The square on the diagonal of one of the square tiles would be seen to contain four of the isosceles right triangles into which that diagonal divided the block, while each of the squares on the sides contained two such equal triangles.* To Pythagoras, however, belongs the merit of having given the first general proof of the proposition—a proof the same, there is no reason to doubt, as that now given in our Geometries. Many different proofs of this historically interesting proposition, it may be noticed in passing, have been offered; a number of them have been collected in one publication by a German named Hoffman.

Materials do not exist for writing a succinct history of the progress of Mathematics in the century and a half between Pythagoras and Euclid. We know that it was a period in which there was much geometric study, and in which considerable additions were made to the fund of Mathematical knowledge, but of these only meagre and fragmentary records remain. There were three problems which especially engaged the attention of mathematicians:

1. The duplication of the cube.
2. The trisection of an angle.
3. The quadrature of the circle.

—two, three, four, one might say as a mnemonic. The work of some few men—leading like stepping-stones from the one to the other of the great names mentioned—deserves to be noticed, Our account is in the main taken from Marie's "Histoire des Sciences Mathematiques et Physiques." Hippocrates of Chios, (bn. 450 B. C.) wrote a book on the elements of

*See the article--Pythagoras--in the Encyclopædia Britannica, from which many of the above statements are derived.

Geometry; he was the first to reduce the problem of the duplication of the cube to that of the insertion of two mean proportionals between two given magnitudes. This Hippocrates is not to be confused with the physician of the same name.

Zenodorus (bn. 450 B. C.) is the author of the oldest work on Geometry which has come down to us. This work was preserved by Theon, of Alexandria, in his commentary on the "Syntax" of Ptolemy. Zenodorus attempted to combat the opinion, then common, that equal contours enclose equal surfaces.

Archytas (440—380 B. C.), according to Diogenes Laertius, "was the first person who applied mathematical principles to mechanics and reduced them to a system. and the first also who gave a methodical impulse to descriptive geometry in seeking in the sections of a demicylinder for a proportional mean which should enable him to find the double of a given cube." He was also the first person who ever gave the geometrical measure of a cube, as Plato mentions in his Republic.

The great Plato (430 B. C.—347 B. C.) left no work on Geometry, but he rendered mathematics a signal service in directing the attention of his disciples to the study of the conic sections, and in the invention of loci for the solution of the problems mentioned above.

Eudoxus of Cnidus (409 B. C.—356 B. C.) was a man of varied learning. Apollodorus in his "Chronicles" says that "he was the inventor of the theory of crooked lines."* Archimedes, in his letter conveying to Dositheus, his treatise "On the Sphere and Cylinder," attributes to Eudoxus the theorems: "a pyramid is the third part of a prism having the same base and altitude; a cone is the third part of a cylinder having the same base and altitude." Eudoxus gave a solution of problem (1) which Eratusthenes regarded as excellent, but which is lost.

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*Diogenes Laertius, Sub. nomine.

Menaechmus (bn. circa. 395 B. C.) studied particularly the elementary theory of conics. He gave a solution of problem (2) worthy to be mentioned. Using the modern notation it is as follows: Let two parabolas have their axes at right angles, and let p and q be their respective parameters. Then the equations will be $y^2=px$ (1) and $x^2=qy$ (2)

$$\text{from (1) } p:y=y:x$$

$$\text{and from (2) } q:x=x:y;$$

for the point of intersection,

$$p:y=y:x=x:q.$$

So to find the required proportionals between two quantities, with these lines as parameters construct two parabolas with axes at right angles; then the coördinates of the point of intersection will be the required proportionals.

The name of Euclid has become literally synonymous with Elementary Geometry. Of no man can it be more truly said "He lives in his work." Indeed he scarcely lives for us outside of it. His fate is that of some of the world's greatest—to have handed down an utterly imperishable work, and yet to have left upon history but scant impress, if any, of his own personality—the fate, for instance, of Homer and of Shakspere. Euclid's life, falling midway between the age of fable and that of careful and minute historic record, lacks the wealth of legend and tradition with which the stories of Thales and of Pythagoras were richly woven about, and fails of the full narration which it might have received in later years. Nothing is definitely known as to his parentage and place of birth. He flourished in the first half of the third century B. C. Proclus asserts that he was younger than the associates of Plato, but older than Eratosthenes (276—190 B. C.) and Archimedes (287—212 B. C.) The new Egyptian city, Alexandria, sheltering the ashes and perpetuating the name of its great founder, was just rising into importance as a centre of culture.

Ptolemy Soter, to whom Egypt had fallen in the division of Alexander's spoils, had founded the great Alexandrian library, and gathered about him a circle of *savants*. Among others Euclid was invited thither, and here founded his school of mathematics. The King himself—so the story goes—was led by his enthusiasm for learning to become a pupil of the great mathematician, but finding the "Elements" rather more difficult reading that that to which his kingly patience was accustomed, inquired if there was not some easier way of learning the subject. To which Euclid made the celebrated reply which stands at the head of this paper: "There is no royal road to Geometry." Two of Euclid's mathematical works have been preserved, the Elements (Στοιχεῖα) and the Data (Δεδομένα). Euclid's Elements have been accepted in all later times as embodying the essential requirements of primary geometrical teaching. Boetius, senator and philosopher, the last of the Romans of the old school, is said to have translated a part of the Elements into Latin (6th century), but in the lack of consistency among the manuscripts, critics are inclined to doubt their authenticity. The Arabs, to whose labors we are so largely indebted for the preservation of learning during the Dark Ages, busied themselves with translations of Euclid; one such translation by Nasr-ed-Din Ibn-Hassan, the Persian astronomer of the thirteenth century, appeared at Rome in 1594. The first printed edition was a translation from the Arabic by the Italian, Campano, which was made in 1482. About twenty years later a translation from the Greek was made by Zamberti, and printed at Venice. Our one English edition containing all the works of Euclid is the Oxford edition, published by Dr. David Gregory in 1703, with the title *Εὐκλείδου τὰ σωζόμενα*. The compilation which has formed the basis of later English works on the subject, is the one given forth in 1756 by Dr. Robert Simson, Professor of Mathematics in the University of Glasgow. It comprised the first six books of Euclid, some of the eleventh book and two propositions of the twelfth. A favorite text is that of the late Prof. Todhunter, which is founded on Simson's. On the continent Euclid has not been so strictly followed as in Great Britain. Our American treatises on Elementary Geometry are generally modelled after the French, a large number of them being merely adaptations of the work of the amiable and great Legendre.

The "Elements" of Euclid consisted of thirteen books, to which were added two others, on the five regular polyedra, of which the Alexandrian, Hypsicles, is supposed to have been the author. Euclid was both a collector and an originator. No means exist of discriminating exactly the two parts of his work. We have seen that certain propositions had been discovered before him. How many more of those which are gathered in his collection were due to others we cannot say. "Euclid," says Proclus, "put in order many things discovered by Eudoxus, perfected what Theaetaetus had begun, and demonstrated more rigorously what had previously been too loosely proved."

The first book begins with the definitions, the postulates, and the axioms. Here Euclid is laying the foundation of his science, and just here does he meet with the largest amount of cavil on the part of critics. There is certainly ground for objection to some of his statements, but a discussion of them would be out of place here. A domain far beyond the ken of Euclid's restrictions has been glimpsed after two thousand years by the inventions of Hamilton, Grassman, and others.

The first proposition is the problem to describe an equilateral triangle on any sttaight line as a side, and the first theorem is as to the congruency of two triangles which have two sides and the included angle of the one equal to ths corresponding parts of the other. The book ends with the Pythagorean theorem and its converse. The second book treats of the relations between squares and rectangles formed on certain lines and their segments. It contains two problems—to divide a line into extreme and mean ratio, and to describe a square equal to a given rectilinear figure. Stated algebraically, the first problem is to find x so that $a\ (a-x)\times x^2$ or $x^2+ax=a^2$, and so it involved the solution of one form of a quadratic equation. The third book is on the circle. The fourth book consists entirely of problems on the inscription and circumscription of circles and polygons, including the problem to construct an isosceles triangla, having each angle at the base double of the angle at tho vertex, which is used in inscribing a regular pentagon in a circle. The fifth book is devoted to Euclid's celebrated treatment of proportion. The essence of the treatment lies in the definition of proportionality; and its superiority consists in the generality which flows from this definition, and renders the method applicable to incommensurable magnitudes as well as to commensurable. The sixth book contains a number of theorems and problems involving the application of proportion. So far the enunciations are all for figures in one plane. Books seven, eight and nine are on Arithmetic. Book ten is on incommensurables. Books eleven, twelve and thirteen are chiefly on Solid Geometry.

The Data of Euclid comprised, according to Pappus, ninety propositions; in the extant editions ninety-five propositions are included under the designation. Dr. Simson has left an edition of these also. The Data were propositions in which it is required to prove that certain things being given certain others may be determined—that is are *potentially* given, since involved in the hypothesis. The work was intended as a kind of supplement or appendix to the elements, designed to facilitate the application of the principles contained in them to the solution of problems.

As examples, we may cite:

"If from a given point a line is drawn, touching a circle given in position, the line is given in position and magnitude."

And (Prop. 6), "If two quantities are to each other in a given ratio, the quantities compounded of the two shall be to each other in a given ratio."

Among the lost works of Euclid of which we have record are: Two books on Plane Loci, four on Conics, and three on Porisms. Simson thought that the books on Plane Loci treated of curves of double curvature, an opinion which was shared by the historian, Montucla. Chasles, in the introductory lecture delivered upon the inauguration of his course in Higher Geometry at Paris, took the view that they treated of surfaces of revolution of the second degree and the sections of them by planes—with whom M. Marie, in his recent history of Mathematics, agrees.

Pappus says that Euclid wrote four books on Conics which formed the basis of the great work of Apollonius, the "Sublime Geometer." Apollonius in his letter transmitting his treatise to Eudemus, says that in his first four books he had elaborated that which had been done before him, and especially mentions a certain problem which had been solved by Euclid only in a special case. We have, however, no information which enables us to speak with any degree of certainty of the content of Euclid's work.

The Porisms of Euclid present one of the profoundest of mathematical enigmas. What did Euclid mean by a Porism, and what were the propositions which he enunciated under that name? Commentators and editors, among them some of the brightest of geometers, have essayed the solution of this question. Albert Girard, in the first half of the seventeenth century expressed the hope that he might restore the lost Porisms, and Fermat, a little later, touched upon the same subject. In 1776 appeared a posthumous work of Simson's, "*De Porismatibus tractatus; quo doctrinam Porismatum satis explicatam et in posterum ab oblivione tutam fore sperat Auctor.*" In our own century the great Chasles has made a brilliant effort at the re-establishment of the lost work in his "*Trois Livres de Porismes d' Euclide retablis, etc.*"

Our definite information concerning Porisms is found in a notice given by Pappus in the seventh book of his "Mathematical Collection," and a brief mention by Proclus in his commentary on the first book of the elements. Pappus says that the Porisms of Euclid were an ingenious collection of a number of propositions, serving for the solution of the most difficult problems; that the ancients defined Theorem, Problem and Porism as propositions respectively in which it is required to prove, to construct, and to find something. Proclus gives a similar definition of Porisms which, he says, occupy a place intermediate between theorems and problems. Simson defined a porism as "a proposition in which it is required to show that one thing is given, or several things are given, which, as well as any one of an infinite number of other things not given but of which each one bears the same relation to the given things, have a certain common property described in the proposition." Playfair, professor of Mathematics in the University of Edinburgh, in a memoir suggested by Simson's work, defines porism as "a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of an infinite number of solutions." Chasles, after noticing the other attempts at the definition of a porism, presents this one: "Porisms are incomplete theorems expressing certain relations between variables following a common law." Pappus states that there are thirty-eight Lemmas for the three books of Porisms, from which are deduced one hundred and seventy-one theorems.

We believe that this perplexed problem has been still more obscured by the attempts at its elucidation. We believe that those who have attempted to give a definition of "Porism" have been groping around in the dark for what was not there. The name by which Euclid designated the propositions in question

—πόρισμα (gain, extra acquisition)—is the same as that applied to the immediate deductions from theorems which we translate. "Corollary" There is no notice of any definition of Porism given by Euclid; that which Pappus quotes, he attributes to "the ancients." We believe that Euclid in calling these propositions Porisms was not intending to distinguish any essential characteristic resident in all the enunciations, but simply labeling them as, like the Corollaries, "additional propositions"—a "gain" from previously deduced theorems. We do not think that Euclid intended to apply the name Porism to a class of propositions distinct, in some mysterious and hitherto inexplicable way, from both propositions and theorems. No rational explanation of Porisms has ever been offered which did not include them under the one or the other. Pappus, in his notice of them, quoted above, calls them "theorems." Simson says, "A Porism is a proposition in which it is required to *demonstrate*, etc.," and this, according to the definition in Euclid's Elements, certainly constitutes a theorem. Chasles, we have just seen, defines them as "incomplete theorems." The diversity of expression among geometers who have discussed Porisms is due to an effort to frame a definition which shall comply with Pappus' representation of them as different in some way from both theorems and problems, and shall be comprehensive enough to include under it all the cases in question. The probability that Euclid used the word Porism in the sense which we have suggested is increased by the consideration that Diophantus gave the same title to a treatise of his having no connection with geometry, and to which accordingly the definitions of Porisms ordinarily given could not apply. A writer on the subject, speaking of the work just cited, says: "These propositions are not, however, all similar in form, and we cannot by means of them grasp what Diophantus understood to be the nature of a porism." Is is not probable that these were simply additional propositions suggested by the line of thought contained in his great Arithmetic, and that he meant no more in calling them porisms?

In subject matter the Porisms were what Euclid might have called a "Modern Higher Geometry;" they contained, among other things, the elements of the theory of transversals and some properties relative to the anharmonic ratio of four points, and thus were an anticipation of what is known to-day as Modern Higher Geometry.

The following are some of the authorities for the period treated of:

Diogenes Laertius' Lives of Philosophers--written in second century. Uncritical in spirit and unscientific in method, but interesting in matter.

Pappus. Lived at Alexandria in fourth century. His "Collection" in eight books gives results attained by previous mathematicians, together with original discoveries. Very important.

Proclus. The Neo-Platonist, bn. at Constantinople, 412 A. D. Studied at Alexandria. Wrote a commentary on first book of Euclid's Elements.

Ueberweg's History of Philosoph. §§12, 16.

Grote's History of Greece, Vol. II., c. 37.

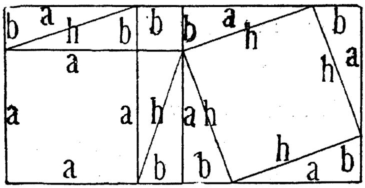
Marie's Histoire des Sciences Mathematiques et Physiques, Vol. I. Recently completed in twelve volumes. (Gauthier-Villars, Paris.)

Chasles' Les trois Livres de Porismes d' Euclide re'tablis. (Mallet-Bachelier, Paris.)

Encyclopædia Britannica, articles, Thales, Pythagoras, Geometry, Porism.

A NEW ELEMENTARY DEMONSTRATION OF THE PYTHAGOREAN THEOREM.

BY DR. WILLIAM B. SMITH, COLUMBIA, MO.



From each of two congruent squares cut away four congruent right triangles; of the one there is left the square on the hypoteneuse of the right triangle; of the other, two squares on the legs of the right triangle; hence the truth of the proposition.*

---

*[The above demonstration of the Pons Asinorum is so good and simple that it is difficult to believe it new. We are inclined to think it is, if for no other reason than that Todhunter in his Edition of Euclid's Elements in remarking on the Theorem in the notes, gives there as the most interesting of the many demonstrations one in which any two unequal squares are used and the proof is not so good as the above.

The largest collection of demonstrations of this proposition seems to be a dissertation by Joh. Jos. Ign. Hoffmann, entitled "Der Pythagorische Lehrsatz . . . Zweyte . . . Ausgabe. Mainz, 1821. This we have not been able to examine.--(Eds.)]

SOLUTIONS OF EXERCISES

1.

Two vertices A and B of a triangle A B C describe straight lines which meet at the angle ω; show that the area of the curve described in their plane by the vertex C is

$$\Omega=\tfrac{1}{2}\pi\,(a^2+b^2+c^2-4\triangle \text{ ct }\omega).$$

$\triangle$ being the area of the triangle A B C. [*W. H. Echols.*]

SOLUTION.

Let the paths of A and B meet in I.

In any position of A B draw the cir-cum circle A B I centered at O whose radius is r. Put C O$=\delta$.

Then the path of C is an ellipse whose semi-axes are $\delta+r$ and $\delta-r$. (Sc. B., Vol. I., No. I.)

Join O A and OB, let O A B$=\alpha$.

Then $$\alpha+\omega=\tfrac{1}{2}\pi.$$

The triangle O C A gives

$$\begin{aligned}\delta^2 &= r^2+b^2-2rb\text{ co}\,(\text{A}+\alpha),\\ &= r^2+b^2-2rb\text{ co}\,(\tfrac{1}{2}\pi+\text{A}-\omega),\\ &= r^2+b^2-2rb\text{ si}\,(\text{A}-\omega).\end{aligned}$$

The triangle O C B gives in like manner,

$$\delta^2 = r^2+a^2-2ra\text{ si}\,(\text{B}-\omega).$$

Whence results

$$\begin{aligned}2(\delta^2-r^2) &= a^2+b^2-2r[b\text{ si}\,(\text{A}-\omega)+a\text{ si}\,(\text{B}-\omega)],\\ &= a^2+b^2\\ &\quad -2r[\text{co }\omega\,(b\text{ si A}+a\text{ si B})-\text{si }\omega\,(b\text{ co A}+a\text{ co B})],\\ &= a^2+b^2-\frac{\text{c}}{\text{si }\omega}[2h\text{ co }\omega-\text{c si }\omega].\end{aligned}$$

Since $c=2r \text{ si } \omega$, $b \text{ co } A+a \text{ co } B=c$, $b \text{ si } A=a \text{ si } B=h$.

Hence $$\Omega=\pi(\delta+r)(\delta-r)=\pi(\delta^2-r^2),$$
$$=\tfrac{1}{2}\pi\left\{a^2+b^2-\frac{c}{\text{si }\omega}[2h \text{ co } \omega-c \text{ si } \omega]\right\}$$
$$=\tfrac{1}{2}\pi(a^2+b^2+c^2-2ch \text{ ct } \omega),$$
$$=\tfrac{1}{2}\pi(a^2+b^2+c^2-4\triangle \text{ ct } \omega).$$

Since ch is double the area of A B C. [*W. H. Echols.*]

3.

Two parallel straight lines are distant apart d; it is required to unite them by *two* circular arcs *of given radii* which shall have between them a common tangent of length t.

[*Elmo G. Harris.*]

SOLUTION.

Let L be (the length of the cross-over) the distance between the points of contact with the parallel tangents measured parallel to them.

Let R and r be the radii. Join the centers of the circles and call α the compliment of the angle which this line makes with t.

Then $$\text{ta } \alpha=\frac{t}{R+r}.$$

It is easy to see that the central angles of the two arcs are equal, each represented by δ, say.

Then $$L=(R+r) \text{ si } \delta+t \text{ co } \delta, \qquad (1)$$
$$d=(R+r)(1-\text{co } \delta)+t \text{ si } \delta, \qquad (2)$$
or $$(d-R-r)=t \text{ si } \delta-(R+r) \text{ co } \delta. \qquad (3)$$

Square (1) and (3), add them and reduce the result to
$$L^2-t^2=2d(R+r)-d^2.$$

This gives the relation between L and t, either may therefore be furnished with the data.

Also
$$\text{ta}\,(\alpha+\delta)=\frac{L}{R+r-d},$$
$$=\frac{\text{ta}\,\alpha+\text{ta}\,\delta}{1-\text{ta}\,\delta\,\text{ta}\,\alpha}.$$
Hence
$$\text{ta}\,\delta=\frac{L\,(R+r)-t}{R+r+t\,L}.$$

This solves the problem. If the radii are equal we have the familiar railway engineers' cross-over, and the results are
$$L^2-t^2=4d\,R-d^2,$$
$$\text{ta}\,\delta=\frac{2R\,L-t}{2R-t\,L}.$$

[*Elmo G. Harris.*]

[*Also by W. O. Whitescarner and Charles Puryear.*]

5.

Two straight lines O P and O Q are of lengths b' and a' respectively. From P a perpendicular P M is drawn to O Q and equal to it, cutting it at N. Show that the equation to the locus of P, as the point N moves on O Q and the point M on Q M, referred to O Q and O P as axes of x and y respectively, is
$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$
[*W. H. Echols.*]

SOLUTION.

Let P′ N′ M′ be any position of the moving line. Let the angle between M O produced and O Q be ω. Find the equation to the locus of P′ referred to these as axes of y' and x' respectively. Thus, drawing the ordinate P′ A$=y'$, the triangles O N′ M′ and N′ P′ A give
$$\frac{O\,N'+N'\,A}{N'\,A}=\frac{M'\,N'+N'\,P}{N'\,P'},$$
or
$$N'\,A=\frac{b'\,x'\,\text{si}\,\alpha}{a'},$$

where $\alpha = \text{P O Q}.$

The triangle N′ P′ A gives

$$\text{N}'\text{P}'^2 = b'^2 \text{ si}^2\alpha = y'^2 + \text{N}'\text{A}^2 + 2\text{N}'\text{A}\, y \text{ co } \omega,$$

$$= y'^2 + \frac{b'^2\, x'^2 \text{ si}^2\alpha}{a'^2} + \frac{2b' \text{ co } \omega \text{ si } \alpha\;\; x'\, y'}{a'},$$

or

$$y'^2 + \frac{b'^2 \text{ si }^2\alpha}{a'^2} x'^2 + \frac{2b' \text{ co } \omega \text{ si } \alpha}{a'} x' y' = b'^2 \text{ si}^2\alpha,$$

as the equation to the locus referred to O Q and O M. Transform this to the axes O P and O Q as y and x axes, by the transformation formulae

$$x' = x + y\frac{\text{si }(\omega - \alpha)}{\text{si } \omega}, \qquad y = y\frac{\text{si } \alpha}{\text{si } \omega},$$

and divide through by si $^2\alpha$.

Thus the equation to the locus is

$$a'^2\, b'^2 = \frac{a'^2}{\text{si}^2\omega} y^2 + b'^2 \left[x^2 + \frac{\text{si }^2(\omega - \alpha)}{\text{si}^2\omega} y^2 + 2\frac{\text{si }(\omega - \alpha)}{\text{si } \omega} x\, y\right]$$

$$+ 2a'b' \text{ co } \omega \left(\frac{x\, y}{\text{si } \omega} + y^2\, \frac{\text{si }(\omega - \alpha)}{\text{si}^2\omega}\right).$$

$$2 \text{ area O P M} = a'\, b' \text{ co } \alpha,$$

also

$$= \text{O M}\cdot\text{OP si } (\pi - \omega + \alpha),$$

or

$$= -\frac{b'^2 \text{ co } \alpha \text{ si } (\pi - \omega + \alpha)}{\text{co } \omega}.$$

Therefore

$$\text{si } (\omega - \alpha) = -\frac{a'}{b'} \text{ co } \omega.$$

Substituting this value in the equation above, it reduces readily to the required form

$$a'^2\, y^2 + b'^2\, x^2 = a'^2\, b'^2.$$

[*Charles P. Echols.*]

6.

Regarding the portion of the tangent to the hyperbola intercepted by the asymptotes as one diagonal of a square, what are the loci of the extremities of its other diagonal?

[*W. H. Echols.*]

SOLUTION.

Consider the point on the convex side of the curve, its distance from the tangent is evidently equal to the semi-diameter conjugate to that drawn through the point of contact (x', y') of the tangent to the hyperbola.

Let γ be the angle between these conjugate diameters of the hyperbola, whose center is O, and ρ the distance of the point whose locus is sought from O. Refer the locus to the axes or the hyperbola as coördinate axes.

The equation to the hyperbola is

$$b^2\,x'^2 - a^2\,y'^2 = a^2\,b^2. \tag{1}$$

The relations between the diameters are

$$\left.\begin{aligned} a'^2+b'^2 &= a^2+b^2, \\ a'\,b' \text{ si } \gamma &= a\,b. \end{aligned}\right\} \tag{2}$$

From the triangle (o o, $x\,y$, $x'\,y'$) we have

$$\rho^2 = x^2+y^2 = a'^2+b'^2-2a'\,b' \text{ si } \gamma,$$

$$\therefore\ x^2+y^2 = a'^2+b'^2-2a\,b. \quad - \ - \ - \ - \ - \ - \tag{3}$$

Combining (1), (2) and (3) we have

$$y'^2 = \frac{b^2}{2(a^2+b^2)}\,[x^2+y^2-(a-b)^2],$$

$$x'^2 = \frac{a^2}{2(a^2+b^2)}\,[x^2+y^2+(a+b)^2].$$

But $$(x-x')^2+(y-y')^2 = b'^2,$$

and $$x'^2+y'^2 = a'^2.$$

Hence $$2x\,x'+2y\,y' = x^2+y^2+a^2-b^2.$$

Which squared gives

$$x^2 x'^2+y^2-y'^2-\tfrac{1}{4}(x^2+y^2+a^2-b^2)^2=-2xy\,x'y'. \quad (4)$$

The equation of the normal is

$$a^2 x y'+b^2 y x'=(a^2+b^2)\,x'y',$$

which when squared is

$$a^4 x^2 y'^2+b^4 y^2 x'^2-(a^2+b^2)^2 x'^2 y'^2=-2a^2 b^2 x y x' y'. \quad (5)$$

Combining (4) and (5) to eliminate $x y x' y'$, and substituting in the resulting equation the values for x'^2 and y'^2 as obtained above we readily reduce the equation of the locus

$$\frac{x^2}{(a-b)^2}-\frac{y^2}{(a+b)^2}=1.$$

In like manner the equation to the locus of the other extremity of the diagonal would have been found to be

$$\frac{x^2}{(a+b)^2}-\frac{y^2}{(a-b)^2}=1.$$

[*W. H. Echols.*]

EXERCISES.

7.

On the sides of a triangle T, equilateral triangles are described, all outwards or all inwards. We thus get two new triangles T_1, T_2. Show that

(1). $$\Delta_1+\Delta_2=5\Delta,$$

where Δ, Δ_1, Δ_2 are the areas.

(2). The maximum inscribed ellipses of T_1 and T_2 are confocal.

[*Frank Morley.*]

8.

In the Cassinian $r r_1 = k^2$ the angle between the central radius and one focal radius is equal to that between the other focal radius and the normal.

[*Frank Morley.*]

9.

Solve the equations

$$x^2+yz=ax+bc,$$
$$y^2+zx=by+ca,$$
$$z^2+xy=cz+ab.$$

[*Frank Morley.*]

10.

A 100 foot steel tape is longer than standard, so that at a certain temperature the tape measures a horizontal chord of 100 standard feet under a pull of 16 pounds supported at its ends. Find the pull that will give 40, 50 and in general D (<100) standard foot horizontal chords, at same temperature, when the tape is supported at each end of the 40, 50, D foot graduations.

[*W. O. Whitescarver.*]

11.

A particle is set free at the highest point of a smooth sphere which stands on a horizontal plane. The particle slightly disturbed begins to move in a certain direction, where does it meet the plane and what is the duration of motion?

[*Elmo G. Harris.*]

12.

A smooth tube bent to the shape of a semi-ellipse is fixed in a vertical plane, its major axis horizontal, its semi-minor axis upward. A heavy flexible string passing through the tube and hanging ot rest is cut at one end of the tube. What is the velocity of the string as it leaves the tube? [*W. H. Echols.*]

13.

Given on the ground a circular curve of known radius intersecting a given straight line at a given point and given angle; it is required to unite the two by another circular curve of given radius. [*W. H. Echols.*]

14.

Given on the ground a circular curve of known radius intersecting a given straight line at a given point and given angle; it is required to unite the two by another circular curve of given radius in such a manner as to have a common tangent of length t between the curves. [*W. H. Echols.*]

15.

$$\int (a^2-x^2) \text{ arc-cos} \left(\frac{a}{2\sqrt{a^2-x^2}} \right) dx.$$

[*W. H. Echols.*]

SELECTED. CONSTRUCTIVE PROBLEMS IN ELEMENTARY GEOMETRY.

16.

Construct the triangle A B C which is similar to the given triangle L M N and which projects orthogonally upon a plane into the given triangle A′ B′ C′.

17.

Of the three concurrent edges a, b, c of a cube, the orthogonal projections on a plane a', b' of two are known, it is required to construct the projection of the cube.

18.

Of the three concurrent edges a, b, c of a cube, the orthogonal projection on a plane, a' of one and the directions of the projections of the other two are known, it is required to construct the projection of the cube.

19.

Of the three concurrent edges a, b, c of a cube, the orthogonal projection on a plane a' of one, the lengths of the orthogonal projections of the other two are known, it is required to construct the projection of the cube.

20.

Of the three concurrent edges a, b, c of a cube the orthogonal projection on a plane a' of one, the length of the orthogonal projection of another and the direction of the projection of the third is known, it is required to construct the projection of the cube.

CONTENTS.

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IMAGINARIES IN ANALYTIC GEOMETRY.

BY DR. WILLIAM B. SMITH, COLUMBIA, MO.

In Arts. 253—261 and elsewhere in the classic treatise of Salmon on *Conic Sections*, as well as in other works of like scope, will be found a number of analytic properties of the binary quadric, or General Equation of second degree in Cartesian resp. trilinear coördinates, interpreted geometrically in terms of imaginary points and right lines, whether in finity or at infinity. The formal correctness of these interpretations is, of course, not to be questioned, but it is equally manifest that the visible geometric depiction is altogether inadequate to express the relations under consideration.

By use of the quadrantal versor i as an operator, to denote the turning of an ordinate y through a right angle into perpendicularity to the plane of X Y. Mr. Carr, in his *Synopsis of Pure Mathematics*, enlarges measurably the range of geometric representation. Thus the Equation

$$x^2 + y^2 = a^2,$$

for $$-a \leqq x \leqq a,$$

is depicted by a circle of radius a about the origin, the axes being rectangular. For x lying outside of these extremes the value of y is $i\sqrt{x^2-a^2}$, and the geometric picture is accordingly an equiaxal Hyperbola having the same parameter and real axis as the circle, but in a plane normal to X Y along X. If now, for convenience, this Hyperbola be rotated about X into the original plane X Y, whereby the foot of the ordinate y will not be changed, in this new position it is called, following Poncelet, *supplementary* to the circle as *principal*. In general the two curves

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

are principal and supplementary. Manifestly, supplementaries to the same principal will vary with the choice of conjugate diameters for coördinate axes.

With help of the supplementary curve, many puzzling properties of conics, such as

All circles meet in the same two imaginary points at infinity;

Concentric circles touch in four imaginary points at infinity;

All confocal conics have four common tangents imaginary and determining four foci, two real, two imaginary, as two pairs of their opposite intersections;

and the like, may now be interpreted geometrically and visibly.

Two reflections, however, suggest themselves. By turning the perpendicular curve into the plane X Y and then using its properties to supplement the properties of the principal, we seem really to surrender the problem of interpreting our analytic properties through the circle and to say in effect. "We can not understand these of the visible circle, but we may understand them of the visible hyperbola." Accordingly, the problem of rendering these properties, when affirmed of the circle, intelligible to intuition, seems scarcely to have been met and solved but rather evaded. The burden which proved too heavy for the principal shoulder has been shifted over to the supplementary one. Thus, when asked to say in what sense the circle

$$x^2+y^2=a^2$$

has a pair of imaginary asymptotes

$$x^2+y^2=0,$$

we make answer that the supplementary hyperbola

$$x^2+y^2=a^2$$

has a pair of real ones

$$x^2-y^2=0.$$

The answer is indeed quite correct, but not quite relevant.

Again, in dealing with extra-real values of the coördinates, either of two ways seems logically open: to admit all or to admit none. Choosing the latter, we must say of the Asymptotes

$$x^2+y^2=0$$

simply that they *are not*, no finite real values satisfying their Equation; this latter is accordingly a mere analytic symbolism void of geometric content. This answer is entirely correct and consistent, involving no internal contradiction. So with respect to imaginary points of intersection of conics, we may say curtly there *are no such points* and so end the discussion. But if we choose the other path and admit any imaginary values to equal rights with real ones, then we must admit all, "for there is no difference." Any reason which legitimates the value i for y in $x^2+y^2=1$ must legitimate the same value for x and the general value $a+ib$ for both. The fact is, so soon as the ditensive unit i is recognized at all the domain of number becomes a manifold doubly extended and is no longer to be pictured by a continuity of points along a single axis as X or Y, but requires a surface, as a plane, for its complete depiction. Very naturally, then, the geometric interpretation of the Equation in x and y, where each may be of the form $a+ib$, as a curve in the plane X Y, while *quoad* perfect, is yet incomplete, for there is no place on either axis for the geometric picture of an imaginary value, and hence no place in their plane for the picture of a pair of such values. Evidently, then, if we would interpret the Equation completely, we must construct the values of the coördinates completely: we must assume two planes, instead of two axes, of x and y. In each of these planes we assume two axes at right angles, the one of pure reals, the other of pure imaginaries. For convenience, suppose the planes perpendicular to each other, then in general we shall have four mutually perpendicular right lines, which are possible only in at least four-fold space. Such a space, though perfectly reasonable, is not imaginable, our intuition reaching only to three dimensions. Our equation constructed in this space would yield a solid as a border between two four-fold extent, and while amenable to analytic treatment would still defy envisagement as effectually as did our imaginary elements in the original plane. However, there is nothing to prevent our assuming two axes of reals at right angles, and a third axis normal to their plan e at their intersection as the *common axis* of pure imaginaries. If this be named the Z-axis, then the whole domain of value of x will be geometrically the X Z-plane, and of y, the Y Z-plane. Now put

$$x=u+i\,u' \qquad \text{and} \qquad y=v+i\,v';$$

then these two points (u, u'), (v, v'), in X Z and Y Z, are two opposite vertices of a parallelogram, of which the origin and the point (x, y) are the other pair of vertices. The rectangular coördinates of this point (x, y) are plainly u, v, $u'+v'$, or u, v, z, all of which are always real. To any pair (x, y) corresponds a triplet (u, v, z); accordingly the complete depiction of the equation in x and y, when complex values are admitted, will be the perfect depiction of the corresponding equation in u, v, z when only real values are admitted. This latter will of course be a surface in the space of u, v, z. It remains to transform the Equation in x, y into an equation in u, v, z.

The General Equation of first degree in x and y is

$$lx+my+n=0; \tag{1}$$

whence

$$l(u+iu')+m(v+iv')+n=0; \tag{2}$$

whence

$$lu+mv+n=0, \tag{3}$$

and

$$lu'+mv'=0 \tag{4}$$

Equation (3), independent of z, is that of a plane parallel to Z, its trace on the plane U V being the right line

$$lu+mv+n=0.$$

Equation (4) does not limit in any way the total locus of (3), but merely declares how the z of each point of the locus is set together out of u' and v'. Here the constants l, m, n have been supposed real, as is uniformly done in discussions of this equation. But that supposition is by no means a necessary one. If we attribute to them the most general values,

$$a+ia', \quad b+ib', \quad c+ic',$$

then result the equations

$$\begin{aligned} au+bv+c-a'u'-b'v'&=0 \\ a'u+b'v+c'+au'+bv'&=0 \quad ; \qquad \text{also,} \\ z-u'-v'&=0. \end{aligned}$$

whence, eliminating u' and v', we have

$$\begin{vmatrix} au+bv+c, & -a', & -b', \\ a'u+b'v+c', & a, & b, \\ z, & -1, & -1, \end{vmatrix}=0, \quad \text{or}$$

$$(a^2-ab+a'^2-a'b')u-(b^2-ab+b'^2-a'b')v+(a'b-ab')z \\ +ac-bc+a'c'-b'c'=0,$$

the equation of a plane not in general perpendicular to the original plane U V. Examples of such oblique planes will hereafter present themselves. Omitting at this point further discussion of this equation, let us pass to Equations of second degree. Of these the simplest is that of the circle about the origin,

$$x^2+y^2=a^2.$$

Or $$(u+i\,u')^2+(v+i\,v')^2=a^2; \tag{5}$$

whence $$u^2+v^2-(u'^2+v'^2)=a^2, \tag{6}$$

and $$u\,u'+v\,v'=0. \tag{7}$$

From (7) and $u'+v'=z$ there result

$$v'=\frac{u\,z}{u-v}, \qquad u'=-\frac{v\,z}{u-v}, \tag{8}$$

whence, on substitution in (6) and reduction,

$$(u^2+v^2)\,[(u-v)^2-z^2]-a^2(u-v)^2=0, \tag{9}$$

a surface of fourth degree.

To get a clearer idea of this quartic let us introduce polar coördinates by the relations

$$u=\rho\,\underline{\vartheta},\quad v=\rho\,\vartheta_{|},$$

where $\underline{\vartheta}$ and $\vartheta_{|}$ stand for cosine and sine of ϑ, the inclination of ρ from U-axis. Then (9) becomes, on rejecting ρ^2,

$$(\rho^2-a^2)\,(\underline{\vartheta}-\vartheta_{|})^2-z^2=0. \tag{10}$$

Here $(\underline{\vartheta}-\vartheta_{|})^2$ is a pure number positive and constant for ϑ constant; call it k^2; then, taking ρ and z as coördinates of the curve of section of the surface with the plane through Z sloped ϑ to U, we have, as its rectangular equation,

$$\frac{\rho^2}{a^2}-\frac{z^2}{a^2\,k^2}=1, \tag{11}$$

an Hyperbola, or, for varying ϑ, a family of Hyperbolas. The *parameter of this family* is k^2; the real axes are all $2a$ and form the pencil of diameters of the circle

$$u^2+v^2=a^2;$$

for $\vartheta = 0$ the conjugate axis is $2a$, the hyperbola is equilateral; as ϑ increases to $\frac{1}{4}\pi$ the conjugate axis shrinks to 0, the hyperbola flattens to a doubly laid right line bisecting outside of the circle the angle U O V; as ϑ goes on increasing to $\frac{1}{2}\pi$, k^2 passes through the same system of values, yielding the same system of hyperbolas, in opposite order; for ϑ increasing to $\frac{3}{4}\pi$, k^2 rises to its maximum, 2, thence for ϑ increasing to π it once more sinks through opposite stages to its original value, 1. Herewith the circuit of its values is complete and is merely repeated as ϑ passes from π to 2π. Accordingly, as a plane turns about z it cuts the surface continually in an Hyperbola, with vertex on the circle

$$u^2+v^2=a^2,$$

with constant real axis $2a$, and with conjugate axis ranging continuously from 0 for $\vartheta=\frac{1}{4}\pi$ or $\frac{5}{4}\pi$ to $2a\sqrt{2}$ for $\vartheta=\frac{3}{4}\pi$ or $\frac{7}{4}\pi$. The surface is symmetric with respect to two planes bisecting the angles of the real axes, U and V, as becomes analytically clear on turning the axes through an angle, $\frac{1}{4}\pi$; it consists of two halves compendent along the inner bisector, $u=v$.

Now suppose $a=0$; the circle reduces to the point-circle

$$x^2+y^2=0,$$

which is also the pair of imaginary Asymptotes

$$(x+iy)(x-iy)=0.$$

But $a=0$ reduces our equation (9) to

$$u^2+v^2=0 \quad \text{or} \quad (u-v)^2-z^2=0. \tag{12}$$

Of these the first is pictured completely by the origin, since u and v are expressly real, the second breaks up into the two disjunctive equations

$$u-v-z=0 \quad \text{and} \quad u-v+z=0. \tag{13}$$

These are depicted by two planes meeting on the right line $u-v=0$, bisecting the coördinate angle U O V; the first also bisects the angle V O Z; *i. e.*, the planes bisect the coördinate angle U O V of the *original* axes and are inclined, each at an angle whose tangent is 2, to the plane of those axes. Also, the plane turning about z cuts this pair of planes in the curve

$$k^2\rho^2-z^2=0, \qquad (14)$$

i. e., in the pair of right lines

$$k\rho-z=0 \quad \text{and} \quad k\rho+z=0. \qquad (15)$$

But these right lines are plainly the Asymptotes to the section of the surface made by the rotating plane, namely, to the Hyperbola

$$k^2\rho^2-z^2=k^2a^2. \qquad (16)$$

Hence we see that the locus

$$x^2+y^2=0$$

is genuinely and completely asymptotic to the locus

$$x^2+y^2=a^2,$$

every rectilinear radial section of the first being asymptotic to the corresponding hyperbolic section of the second.

Now change the sign of a^2; then results the purely imaginary circle

$$x^2+y^2=-a^2.$$

Its complete spatial depiction is obtained at once by changing the sign of a^2 in the foregoing reasoning; the asymptotic planes are unaffected; while all the hyperbolas pass over into their conjugates. Thus the imaginary circle stands to the real circle, not only analytically but also visually, precisely as the conjugate hyperbola stands to its primary, the one being quite as "real" as the other, and both having the common real asymptotic planes

$$x^2+y^2=0.$$

The complete spatial depiction of the real and (so-called) imaginary ellipses,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm 1,$$

is now easily apparent. It suffices to supplace y with $\frac{b}{a}y$ in in the foregoing. The general relations remain undisturbed.

Let us now pass to the rectangular hyperbola,

$$x^2-y^2=a^2 \qquad (17)$$

Proceeding as in case of the circle we obtain

$$(u^2-v^2)\,[(u+v)^2+z^2]-a^2(u+v)^2=0, \qquad (18)$$

or

$$(\underline{\vartheta}+\vartheta_1)^2\,\rho^2+z^2=a^2(\underline{\vartheta}+\vartheta_1)^2(\underline{\vartheta}^2-\vartheta_1^2), \qquad (19$$

an Ellipse in the plane through Z turned ϑ from U, with center at origin, one axis $2a/\sqrt{\underline{\vartheta}^2-\vartheta_1^2}$, the other $2a(\underline{\vartheta}+\vartheta_1)/\sqrt{\underline{\vartheta}^2-\vartheta_1^2}$. For $\vartheta=0$ this ellipse becomes a circle with diameter $2a$; as ϑ increases both axes increase, the second the faster, which is therefore the axis major, until for $\vartheta=\frac{1}{4}\pi$ both become infinite. For ϑ ranging from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$ the sections are strictly imaginary ellipses, since both ρ and z are expressly real; *i. e.*, no part of the real surface lies in this quadrantal region. As ϑ increases from $\frac{3}{4}\pi$ to π, the section, once more real, shrinks from an infinite ellipse to the initial circle, radius a; and herewith the circuit of values is complete, to be retraced as ϑ ranges from π to 2π. The minor axes of all these real elliptic sections are the primary diameters of the hyperbola under consideration. The Asymptotes

$$x^2-y^2=0=(x-y)\,(x+y)$$

are two planes through Z bisecting outerly and innerly the original coördinate angles and touching the surface at infinity all around along the infinite ellipse.

Now change the sign of a^2; the hyperbola passes over into its conjugate

$$x^2 - y^2 = -a^2;$$

all the elliptic sections which were real for a^2+, namely, all for ϑ ranging from $-\frac{1}{4}\pi$ to $+\frac{1}{4}\pi$, now become imaginary, while all which were imaginary, namely, for ϑ ranging from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$, now become real; and the Asymptotes remain the same.

It is hardly necessary to detain the reader with further exemplifications. It seems entirely evident that the so-called imaginary points, lines, circles, ellipses, yea, curves and properties in general, are no longer imaginary in the *lucus a non lucendo* sense of *un*imaginable, but that they exist for the spatial imagination altogether as genuinely as any of the reals of Analytic Geometry.

Further discussion is reserved for the present.

Columbia, Mo., Aug. 11th, 1890.

OUR BELIEF IN AXIOMS, AND THE NEW SPACES.

By Dr. GEORGE BRUCE HALSTED, University of Texas.

"Prove all things, hold fast that which is good," does not mean demonstrate everything. From nothing assumed, nothing can be proved. "Geometry without axioms," was a book which went through several editions, and still has historical value. But now a volume with such a title would, without opening it, be set down as simply the work of a paradoxer.

The set of axioms far the most influential in the intellectual history of the world was put together in Egypt: but really it owed nothing to the Egyptian race, drew nothing from the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that Geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoön.

In a geometry occur the most noted set of axioms, the geometry of Euclid, a pure Greek professor at the University of Alexandria.

Not only at its very birth did this typical product of the Greek genius assume sway as ruler in the pure sciences, not only does its first efflorescence carry us through the splendid days of Theon and Hypatia, but unlike the latter, fanatics cannot murder it; that dismal flood, the dark ages, cannot drown it. Like the phœnix of its native Egypt, it rises with the new birth of culture. An Anglo-Saxon, Adelard of Bath, finds it clothed in Arabic vestments in the land of the Alhambra. Then clothed in Latin, it and the new-born printing press confer honor on each other. Finally back again in its original Greek, it is published first in Queenly Venice, then in stately Oxford, since then everywhere. The latest edition in Greek is just issuing from Leipsic's learned presses.

How the first translation into our cut-and-thrust, survival-of-the-fittest English was made from the Greek and Latin by Henricus Billingsly, Lord Mayor of London, and published with a preface by Jonh Dee the Magician, may be studied in the Library of our own Princeton College where they have, by some strange chance, Billingsly's own copy of the Latin version of Commandine bound with the Editio Princeps in Greek and enriched with his autograph emendations. Even to-day in the vast system of examinations set by Cambridge, Oxford, and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms.

The American ideal is success. In twenty years the American maker expects to be improved upon, superseded. The Greek ideal was perfection. The Greek Epic and Lyric poets, the Greek sculptors, remain unmatched. The axioms of the Greek geometer remained unquestioned for twenty centuries.

How and where doubt came to look toward them is of no ordinary interest, for this doubt was epoch making in the history of mind.

Among Euclid's axioms was one differing from the others in prolixity, whose place fluctuates in the manuscripts, and which is not used in Euclid's first twenty-seven propositions. Moreover it is only then brought in to prove the inverse of one of these already demonstrated.

All this suggested, at Europe's renaissance, not a doubt of the axiom, but the possibility of getting along without it, of deducting it from the other axioms and the twenty-seven propositions already proved. Euclid demonstrates things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third. Yet when he has perfectly proved that lines making with a transversal equal alternate angles are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy postulate or axiom;

"If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles."

Do you wonder that succeeding geometers wished by demonstration to push this unwieldy thing from the set of fundamental axioms.

Numerous and desperate were the attempts to deduce it from reasonings about the nature of the straight line and plane angle. In the "Encyclopœdie der Wissenschaften und Künste; Von Ersch und Gruber;" Leipzig, 1838; under "Parallel," Sohncke says that in mathematics there is nothing over which so much has been spoken, written, and striven, as over the theory of parallels, and all, so far, (up to his time) without reaching a definite result and decision.

Some acknowledged defeat by taking a new definition of parallels, as for example the stupid one, "Parallel lines are everywhere equally distant," still given on page 33 of Schuyler's Geometry, which that author, like many of his unfortunate prototypes, then attempts to identify with Euclid's definition by pseudo-reasoning which tacitly assumes Euclid's postulate, e. g. he says p. 35; "For, if not parallel, they are not everywhere equally distant; and since they lie in the same plane, must approach when produced one way or the other; and since straight lines continue in the same direction, must continue to approach if produced farther; and if sufficiently produced, must meet." This is nothing but Euclid's assumption, diseased and contaminated by the introduction of the indefinite term "direction."

How much better to have followed the third class of his predecessors who honestly assume a new axiom differing from Euclid's in form if not in essence. Of these the best is that called Playfairs; "Two lines which intersect cannot both be parallel to the same line."

The German article mentioned is followed by a carefully prepared list of ninety-two authors on the subject. In English an account of like attempts was given by Perronet Thompson, Cambridge, 1833, and is brought up to date in the charming volume, "Euclid and his Modern Rivals," by C. L. Dodgson, late Mathematical Lecturer of Christ Church, Oxford.

All this shows how ready the world was for the extraordinary flaming-forth of genius from different parts of the world which was at once to overturn, explain, and remake not only all this subject but as consequence all philosophy, all ken-lore. As was the case with the discovery of the Conservation of Energy, the independent irruptions of genius, whether in Russia, Hungary, Germany or even Canada gave everywhere the same results.

At first these results were not fully understood even by the brightest intellect. Thirty years after the publication of the book he mentions, we see the brilliant Clifford writing from Trinity College, Cambridge, April 2, 1870, "Several new ideas have come to me lately: First I have procured Lobatchewsky, 'E'tudes Géométriques sur la Theorie des Parallels' - - - a small tract of which Gauss, therein quoted, says: L' auteur a traité la matière en main de maitre et avec le véritable esprit geometrique. Je crois devoir appeler votre attention sur ce livre, dont la lecture ne pent manquer de vous causer le plus vif plaisir'. Then says Clifford: "It is quite simple, merely Enclid without the vicious assumption, but the way the things come out of one another is quite lovely."

The first axiom doubted is called a "vicious assumption," soon no man sees more clearly than Clifford that all are assumptions and none vicious. He had been reading the translation by Hoüel, published in 1866, of a little book of 61 pages published in 1840 in Berlin under the title Geometrische Untersuchungen zur Theorie der Parallellinien by a Russian, Nicolaus Ivanovitch Lobatchewsky, (1793-1856), the first public expression of whose discoveries, however, dates back to a discourse at Kasan on February 12, 1826.

Under this commonplace title who would have suspected the discovery of a new space in which to hold our universe and ourselves.

A new kind of universal space; the idea is a hard one. To name it, all the space in which we think the world and stars live and move and have their being was ceded to Euclid as his by right of pre-emption, description and occupancy; then the new space and its quick-following fellows could be called Non-Euclidean.

Gauss in a letter to Schumacher dated Nov. 28, 1846, mentions that as far back as 1892 he had started on this path to a new universe. Again he says: "La Géometrie non-Euclidienne ne renferme en elle rien de contradictoire, quoique, à première vue, beaucoup de ses resultats aient l'air de paradoxes. Ces contradictions apparents doivent etre regardées comme l'effet d'une illusion, due à l'habitude que nous avons prise de bonne heure de considérer la géométrie Euclidienne comme rigourous."

But here we see in the last word the same imperfection of view as in Clifford's letter. The perception has not yet come that though the non-Euclidean geometry is rigorous, Euclid is not one whit less so.

A clearer idea here had already come to the former room-mate of Gauss at Göttingen, the Hungarian Wolfgang Bolyai. His principal work, published by subscription, has the following title:

Tentamen Juventutem studiosam in elementa Matheseos purae, elementaris ac sublimioris, methodo intuitiva, evidentique huic propria, introducendi. Tomus Primus, 1832; Secundus, 1833. 8o. Maros-Vàsàrhelyini.

In the first volume with special numbering, appeared the celebrated Appendix of his son Johann Bolyai with the following title:

Ap., scientiam spatii *absolute veram* exhibens: a veritate aut falsitate Axiomatis XI Euclidei (a priori haud unquam decidenda) independentem. Auctore Johanne Bolyai de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Captaneo. Maros-Vàsàrhely., 1832. (26 pages of text).

This marvellous Appendix has been translated into French, Italian and German.

In the title of Wolfgang Bolyai's last work, the only one he composed in German, (88 pages of text, 1851,) occurs the following:

"Und da die Frage, *ob zwei von der dritten geschnittene Geraden wenn die Summa der inneren Winkel nicht=2R, sich schneiden oder nicht?*, niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhængige Geometrie abzusondern, und eine auf die Ja Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzen auf ein Wink auch in der ersten gültig seien."

The author mentions Lobatchewsky's Geometrische Untersuchungen, Berlin, 1840, and compares it with the work of his son Johann Bolyai, "an sujet duquel il dit: 'Quelques exemplaires de l'onvrage publié ici ont été envoyés à cette époque à Vienne, à Berlin, à Göttingen. . . De Goettingen le géant mathématique, [Gauss] qui du sommet des hauteurs embrasse du meme regard les astres et la profondeur des abimes, a écrit qu'il était ravi de voir exécuté le travail qu'il avait commencé pour le laisser après lui dans ses papiers.' "

Yet that which Bolyai and Gauss, a mathematician never surpassed in power, see that no man can ever do, our American Schuyler, in the density of his ignorance, thinks that he has easily done.

In fact this first of the Non-Euclidean geometries accepts all of Euclid's axioms but the last, which it flatly denies and replaces by its contradictory, that the sum of the angles made on the same side of a transversal by two lines may be less than a straight angle without the lines meeting. A perfectly consistent and elegant geometry then follows, in which the sum of the angles of a triangle is always less than a straight angle, and not every triangle has its vertices concyclic.

Gauss himself never published aught upon this fascinating subject, but when the most extraordinary pupil of his long teaching life came to read his inaugural dissertation before the Philosophical Faculty of the University of Göttingen, from the three themes submitted it was the choice of Gauss which fixed upon the one "Ueber die Hypothesen welche der Geometrie zu Grunde liegen." Gauss was then recognized as the most powerful mathematician in the world.

I wonder if he saw that here his pupil was already beyond him, when in his sixth sentence Riemann says, "therefore space is only a special case of a three-fold extensive magnitude," and continues: "From this, however, it follows of necessity, that the propositions of geometry cannot be deduced from general magnitude-ideas, but that those peculiarities through which space distinguishes itself from other thinkable three-fold extended magnitudes can only be gotten from experience. Hence arises the problem, to find the simplest facts from which the metrical relations of space are determinable—a problem which from the nature of the thing is not fully determinate; for there may be obtained several systems of simple facts which suffice to determine the metrics of space; that of Euclid as weightyest is for the present aim made fundamental. These facts are, as all facts, not necessary, but only of empirical certainty; they are hypotheses. Therefore one can investigate their probability, which, within the limits of observation, of course is very great and after this judge of the allowability of of their extension beyond the bounds of observation, as well on the side of the immeasurably great as on the side of the immeasurably small."

Riemann extends the idea of curvature to spaces of three and more dimensions. The curvature of the sphere is constant and positive, and on it figures can freely move without deformation. The curvature of the plane is constant and zero, and on it figures slide without stretching. The curvature of the two-dimentional space of Lobatchewsky and Bolyai completes the group, being constant and negative, and in it figures can move without stretching or squeezing. As thus corresponding to the sphere it is called the pseudo-sphere.

In the space in which we live, we suppose we can move without deformation. It would then, according to Riemann, be a special case of a space of constant curvature. We presume its curvature null. It would then lie between the sphere and pseudo-sphere. At once the supposed fact that our space does not interfere to squeeze us or stretch us when we move, is envisaged as a peculiar property of our space. But is it not absurd to speak of space as interfering with anything? If you think so, take a knife and a raw potato, and try to cut it into a seven-edged solid.

Farther on in this astonishing discourse comes the epoch-making idea, that though space be unbounded, it is not therefore infinitely great. Riemann says: "In the extension of the space-construction to the immeasurably great, the unbounded is to be distinguished from the infinite; the first pertains to the relations of extension, the latter to the size-relations.

That our space is an unbounded three-fold extensive manifoldness, is an hypothesis, which is applied in each apprehension of the outer world, according to which, in each moment, the domain of actual perception is filled out, and the possible places of a sought object constructed, and which in these applications is continually confirmed. The unboundedness of space possesses therefore a greater empirical centainty than any outer experience. From this however the Infinity in no way follows. Rather would space, if one presumes bodies independent of place, that is ascribes to it a constant curvature, necessarily be finite so soon as this curvature had even so small a positive value. One would, by extending the beginnings of the geodesics lieing in a surface-element, obtain an unbounded surface with constant positive curvature, therefore a surface which in a homaloidal three-fold extensive manifoldness would take the form of a sphere, and so is finite."

THE VOLUME OF THE PRISMOID AND THE CYLINDROID.

By Prof. W. H. ECHOLS, Rolla, Mo.

In estimating the volume of earth work in the construction of Lines of Communication, a particular solid has occurred so frequently that engineers have given it a specific name; the Prismoid.

Whether the word was used to designate a definite geometrical solid prior to its adaption by engineers for that purpose, I have been unable to discover. The solid has been an extremely interesting one to engineers and much has been written by them upon the subject of its volume. No small subject connected with the profession has probably received so much labor and attention as this, in the direction of facilitating the computation of the volumes of these earthwork solids. The impracticability of an exact result so far as designing a mathematical surface which should coincide with the natural surface of the terrain was early recognized, and all efforts in dealing with the solid have been in the direction of offering approximate methods of computation, which yield results, the errors of which lie well within the limits which good practice demands.

In the sense of facilitating the computation of earthwork solids it is not the writer's intention to write in the present paper anything of a practical nature upon the subject chosen, but rather to discuss it from a purely geometrical point of view, believing, however, that such a discussion will contain matter which is not uninteresting to engineers, and which at the same time will be of practical benefit to them, insomuch as it will make more clear the advantages of the best methods now employed in practice for approximating to the volumes.

As to the practical feature of numerically computing the earthwork solids, that subject was shelved some fourteen years ago when Mr. Wellington and Prof. Davis published their works on Railway Earthwork*. It is because of a notice in the Engineering News of date, May 24, 1890, given to the second edition of the latter work, in which the Editor remarks "We are astonished that the author should be so ill-read in the literature of his subject as to state in his preface: '*The result of the prismoidal rule is for the first time obtained by a simple correction, without calculating the mid-section of these troublesome solids.*' By referring to p. 36 of Estimates of Railway Earthwork, by A. M. Wellington, published in 1874, he will find such corrections fully explained; and this was not the first.", that this paper was undertaken, the connection appearing in the sequel.

Both of these gentlemen base their methods of computation upon the same formula which is obtained by each in the same way. The final result reached is the method now employed in practice which in a few words may be expressed as follows: The mean area of the engineering prismoid is the average of its end areas, *corrected when necessary.* This correction is determined in each case, by computing the volume for *three-level* sections by the so-called prismoidal formula, then by the average of end-areas, the difference being the desired correction.

One in looking through engineering works cannot fail to be struck with the variety of definitions given to the prismoid solid, and in how few cases is the solid defined in a manner which fixes it in words which may be taken as a mathematical definition of the solid. As much as has been written about the prismoid in engineering journals in connection with the com-

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*Computation from Diagrams of Railway Earthwork, A. M. Wellington. D. Appleton & Co., (1874.)

Formulae for Railway Earthwork, John W. Davis. New York Gilliss Brothers Pub., [1877.]

putation of earthwork volumes, no fixed mathematical definition of it has been agreed upon, and more or less confusion exists in the minds of engineers as to exactly what a prismoid is, beyond the definition given in the unabridged dictionaries where it is defined to be "a solid somewhat like a prism."

For the purposes of the present paper we shall use a definition for the prismoid which is derived from that given by Henck in his Fieldbook, Edition 1854. Where he says "A prismoid is a solid having two parallel faces, and composed of prisms, wedges and pyramids, whose common altitude is the perpendicular distance between the parallel faces." Let us adhere to this as defining the prismoid proper. More particularly expressed it appears as follows:

Definition:—A prismoid is a solid having two parallel plane polygons for bases, and whose side surface is made up of plane faces (triangles or quadrilaterals) formed by joining corresponding corners of the bases.

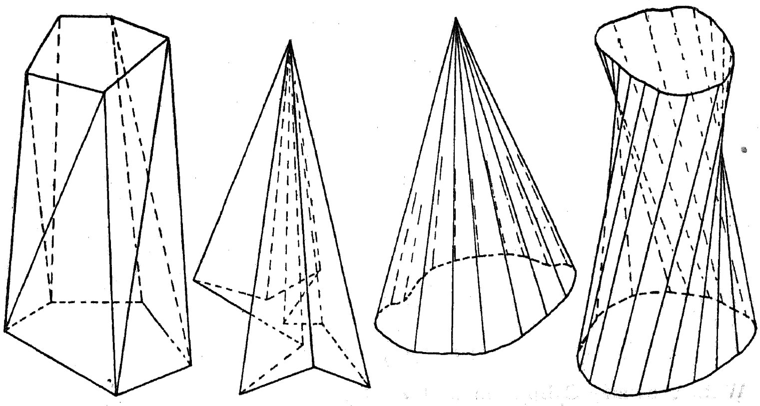
Using *corresponding* corners to denote any two corners, one of each base, such that the straight line joining them is an *edge* of the prismoid.

The property of the first definition follows immediately from the second; that is, it is evident that the solid just defined may be subdivided into prisms, wedges and pyramids; while the second definition serves to give a more definite idea of the shape of the solid as a geometrical figure and leads more directly to what follows below.

The cross-section or simply section of such a prismoid is the section by a plane parallel to the bases. The altitude or length of the prismoid is the perpendicnlar distance between the planes of the bases.

The Associate Pyramid:—If through any fixed point in the plane of one of the bases of the prismoid we draw straights parallel to the lateral edges of the prismoid to meet the plane of its other base in points which are taken to be the corners of a polygon in that plane, these lines define a pyramid whose vertex is in the plane of one base and whose base is in the plane of the other base of the prismoid. This pyramid is called the associate pyramid of the prismoid. It is easy to show that its volume is equivalent to the sum of the volumes of all the component pyramids of the prismoid.

In analogy, with the prism and cylinder of elementary geometry, if about the polygonal bases of the prismoid fixed plane closed curves be circumscribed, we have the following:



Definition:—The Cylindroid is the limit to which the Prismoid approaches when the number of the sides of the inscribed base polygons increases, and their magnitudes decrease, without limit.*

*Wiener in his Lehrbuch der Darstellenden Geometrie, Vol. II, page 471, defines a Cylindroid to be the scroll generated by a straight line guided by a director plane.

"Eine windschiefe Flæche mit einer einzigen, und zwar unendlich fernen Leitgeraden, also mit einer Leitebene, ist das Cylindroid."

In the Theory of Screws, English writers (Ball, Minchin, etc.) apply the name Cylindroid to a particular surface generated by two straights intersecting a third straight in a common point and normal to it, moving along

The plane closed curves become the bases of the cylindroid, and its side surface is a ruled surface. The associate pyramid of the prismoid thus becomes an associate cone to the cylindroid.

Under the above definitions we may now proceed to demonstrate the following theorem, which the writer first heard enunciated by Professor W. M. Thornton, of the University of Virginia, about ten years ago, but which he has never seen in print.

The mean area of the Prismoid (Cylindroid) exceeds the average area of its bases by one-sixth the area of the base of its associate pyramid (cone).**

Considering first the prismoid, we may give here first, the tentative method of identification employed usually to show

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it under fixed law. The angle between the first two straights varying periodically according to law. The equation to the furface is

$$z[x^2+y^2]-axy=0.$$

While the name Cylindroid has been thus differently appropriated to designate these higher mathematical surfaces, it has been thought to be no violation to use it in the present paper for the purpose of clearing up the railroad solid, it being very unlikely that any ambiquity will ever arise.

**In the Third Edition of one of the most recent text books on the Theory and Practice of Surveying we find in a foot note there the nearest approach to this theorem in print. In speaking of the different methods used for computing the earthwork volumes the foot note goes on to say:

"The method by 'mean end areas,' wherein the volume is assumed to be the mean of the end areas into the length, always gives too great a volume (except when a greater center height is found in connection with a less total width, which seldom occurs), the excess being one-sixth of the volume of the pyramids involved in the elementary forms of the prismoid."

This is wrong for the excess is one-half of the volumes of the pyramids involved in the elementary forms of the prismoid.

that Newton's Rule for Mean Area is applicable to the mean area of the prismoid.*

Let B_1, B_2, B_3 represent the area of the base of a component prism, wedge and pyramid of the prismoid respectively. The

volume of prism, $V_1=H[\frac{1}{2}(B_1+B_1)-\frac{1}{6}o]$,

wedge, $V_2=H[\frac{1}{2}(B_2+o)-\frac{1}{6}o]$,

pyramid, $V_3=H[\frac{1}{2}(B_3+o)-\frac{1}{6}B_3]$,

where H is the altitude. Using the same symbols, if B' and B'' are the areas of the bases of the prismoid and B_p that of its associate pyramid, then

$$B'+B''=\Sigma(B_1+B_2+B_3),$$

and

$$B_p=\Sigma B_3.$$

Therefore the volume of the prismoid is

$$V=H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_p].$$

Passing to the limit the volume of the cylindroid is therefore

$$V=H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_c].$$

Using B', B'', B_c to represent the areas of the bases of the cylindroid and its associate cone respectively.

The above is not a demonstration but merely an identification, and is only given here to parallel the process by which

*The so-called demonstration is as follows:

Using the same notations as above, let M be the area of the section mid-way between the bases, then for a component

prism $V_1=HB_1=\frac{1}{6}H[B_1+B_1+4M_1]$, $M_1=B_1$;

wedge $V_2=\frac{1}{2}HB_2=\frac{1}{6}H[B_2+o+4M_2]$, $M_2=\frac{1}{2}B_2$;

pyramid $V_3=\frac{1}{3}HB_3=\frac{1}{6}H[B_3+o+4M_3]$, $M_3=\frac{1}{4}B_3$.

Hence the volume of the prismoid is

$$V=\Sigma(V_1+V_2+V_3)$$
$$=\tfrac{1}{6}H(B'+B''+4M).$$

the result as shown in the foot-note is obtained, which is given by the text books as the prismoidal formula and the demonstration for it.

It would be more logical in giving an elementary deduction of a formula for the prismoid volume to proceed as follows, after first showing that the solid is composed of prisms, wedges and pyramids; thus the volume of a component prism is

$$V=HB_1=\tfrac{1}{2}H2B_1,$$

wedge $$V=\tfrac{1}{2}HB_2,$$

pyramid $$V=\tfrac{1}{3}HB_3.$$

Hence the volume of the prismoid is

$$V=H[\tfrac{1}{2}\Sigma(2B_1)+\tfrac{1}{2}\Sigma B_2+\tfrac{1}{3}\Sigma B_3],$$

$$=H[\tfrac{1}{2}\Sigma(2B_1+B_2+B_3)-\tfrac{1}{6}\Sigma B_3],$$

$$=H[\tfrac{1}{2}(B'+B'')-\tfrac{1}{6}B_p],$$ and so on.

It is only through the direct geometrical process for determining volumes of solids that we arrive in a satisfactory manner at the most appropriate formula for that purpose. Such a formula is then the true one for determining the volume of the particular solid in question as it is in general the simplest one.

Let us regard then the cylindroid as the highest type of the solids we have been considering, of which the prismoid and other degenerate forms are but particular cases. Thus we define the cylindroid independently of the prismoid as follows:

A cylindroid is the solid cut out from between two parallel planes by a moving straight, which finally returns to its initial position.

Alter idem;

A cylindroid is a solid whose bases are two parallel plane surfaces bounded by closed curves, and whose lateral surface is a regulus.

The regulus will in general be a scroll (warped).

Since a scroll is fixed when a linear director, one position of an element and the director cone is given. Then the cylindroid is fixed, given one base, one element and the associate cone, together with the position of the plane of the other base.

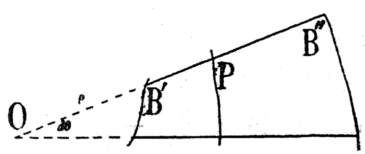
Again, if the cylindroid be given the associate cone is at once fixed, for from thc first definition it follows, that if through a fixed point in one of the planes a straight passes and moves so as to be always parallel to the straight which cuts out the cylindroid, the former cuts out the associate cone from the space between the two parallel planes.

To compute the volume cut from between two parallel planes by a moving straight, we proceed to find first the area of any cross-section (the area of a running section parallel to the base planes).

Project the moving straight and its traces with the planes on any plane parallel to the bases.

Let λ be the length of the projection of that part of the straight which is included between the parallel planes, its extremities being B′ and B″.

Let $d\vartheta$ be the angle through which the line λ turns in making a small shift, O the point of contact of λ with its envelope and ρ the distance of one end of λ from O.



Any plane parallel to the bases cuts the moving straight in a point which divides it in constant ratio, the projection (say P) of this point divides λ in the same ratio (say m/n,) this plane also divides the altitude H of the cylindroid in the same ratio.

Put
$$PB'=\frac{m}{m+n}\lambda=k'\lambda\,; \qquad PB''=\frac{n}{m+n}\lambda=k''\lambda\,.$$

The area swept over by λ is the area included between the two curves traced by its extremities, so that if (B′) and (B″) are the areas of the two closed curves traced, then (B″)—(B′) is the whole area swept over by λ. In like manner (P)—(B′) would be the whole area swept over by $k'\lambda$.

The area swept out by λ turning through $d\vartheta$ is

$$d(\mathrm{B}'')-d(\mathrm{B}')=\tfrac{1}{2}(\rho+\lambda)^2\,d\vartheta-\tfrac{1}{2}\rho^2\,d\vartheta$$
$$=\rho\lambda\,d\vartheta+\tfrac{1}{2}\lambda^2\,d\vartheta\,.$$

But $f(\lambda,\vartheta)=0$ is the polar equation to the base of the associate cone, hence

$$\tfrac{1}{2}\lambda^2\,d\vartheta=d(\mathrm{B_c})$$

and

$$d(\mathrm{B}'')-d(\mathrm{B}')=\rho\lambda\,d\vartheta+d(\mathrm{B_c}) \qquad (1)$$

is the element of area included between the curves traced by B″ and B′ .

In like manner

$$d(\mathrm{P})-d(\mathrm{B}')=\tfrac{1}{2}(\rho+k'\lambda)^2\,d\vartheta-\tfrac{1}{2}\rho^2\,d\vartheta,$$
$$=\rho k'\lambda\,d\vartheta+\tfrac{1}{2}k'^2\lambda^2\,d\vartheta,$$
$$=k'\rho\lambda\,d\vartheta+k'^2d(\mathrm{B_c}) \qquad (2.)$$

Multiplying (1) by k' and substracting (2) from the result, we have, observing that $k'+k''=1$,

$$d(\mathrm{P})=k'd(\mathrm{B}'')+k''d(\mathrm{B}')-k'k''d(\mathrm{B_c}).$$

This is the relation which holds between the elementary areas of the curves traced by the points B′, P and B″, referred to any system of coördinates.

If we integrate for a complete circuit of these points (closed curves) we have the relation between the bases of the cylindroid, the base of its associate cone and any cross-section of the cylindroid parallel to the planes of its bases.

Thus

$$P = k'B'' + k''B' - k'k''B_c.^*$$

Put now $1-k'$ for k'', then

$$P = B' + (B'' - B' - B_c)k' + B_c k'^2 \qquad (a).$$

Thus P is a quadratic function of k'. If now k' be allowed to vary continuously from 0 to 1, then P becomes a running cross-section taking in succession all the values of the sections from one base to the other. The average of all of these is then the mean area of the cylindroid. Thus the mean area of the cylindroid is in symbols

$$\Omega = \frac{1}{1-0}\int_0^1 P\,dk',$$

$$= \int_0^1 [B' + (B'' - B' - B_c)k' + B_c k'^2]\,dk',$$

$$= \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c.$$

Otherwise by the the ordinary geometrical process, let h be the distance of the cross-section P from the plane of one of the bases (say B'), then $h/H = k'$, substituting in (a)

$$P = B' + \frac{B'' - B' - B_c}{H}h + \frac{B_c}{H^2}h^2.$$

The section of a cylindroid is therefore a quadratic function of its length. The volume of the solid is then

$$V = \int_0^H P\,dh.$$

Putting in the second member above for P and operating we have as before

$$V = H[\tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c].$$

This then is the rational formula for computing the volume of any cylindroid or prismoid. It should therefore be expected

*This is Holditch's Theorem.

to give the volume of the solid, with less labor than is required by any more comprehensive formula.

It is easy to eliminate B_c between equations

$$P = k'B'' + k''B' - k'k''B_c$$

and

$$\Omega = \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c,$$

thus getting the mean area Ω in terms of the base areas and that of any cross-section, as for example putting $k' = k'' = \frac{1}{2}$, then P becomes M the mid-section. Hence

$$M = \tfrac{1}{2}(B'' + B') - \tfrac{1}{4}B_c,$$

$$\Omega = \tfrac{1}{2}(B'' + B') - \tfrac{1}{6}B_c.$$

From which by subtraction we see that the mean area differs from the mid-area by one-twelfth the base of the associate cone.

Eliminating B_c we have

$$\Omega = \tfrac{1}{6}(B' + B'' + 4M).$$

This is the form of the so-called prismoidal rule, more generally known as Simpson's Rule, but which is really due to Newton [*Methodus Differentialis*]. It may be found deduced in any good work on Integral Calculus [Todhunter, p. 158]. It is mis-nomer to call it the prismoid formula, for it applies not only to the cylindroid and all of its degenerate forms but applies as well to a large class of solids of a higher order. One would be as well justified in calling the formula above deduced for the cylindroid the "conical formula" because it happens to give the volume of a cone-frustum, as calling Newton's Rule for mean area the "prismoid formula" because it gives the mean area of the prismoid which is only one of the degenerate forms to which the rule applies.

We have seen above that it is a characteristic property of the cylindroid (prismoid), that its cross-sectional area is a quadratic function of its length, therefore the formula for the mean area of the cylindroid is also the formula which gives the mean area of any solid whose cross-sectional area is a linear function of its length, but the converse is not true, for in such solids as the latter (wedges and conoids) there is no associate cone.

Newton's Rule for mean area gives not only the mean area for solids whose sectional areas are linear and also quadratic functions of their lengths, but also of all solids whose sections are *cubic* functions of their lengths. It is therefore far more comprehensive than the true prismoid formula. The demonstration of this is also of the nature of an identification; it is as follows: (Todhunter Int. Cal., p. 173).

Let there be a solid such that the area of a section made by a plane parallel to a fixed plane and at a distance l from it is always

$$P=a+bl+cl^2+dl^3, \qquad (1)$$

where a, b, c and d are constants.

Let three equidistant sections of the solid B′, M. B″ be made by the fixed plane and two others parallel to it in order. Then the volume of the portion of the solid included between the two extreme sections is

$$V=\int_0^L P\,dl,$$

$$=aL+\tfrac{1}{2}bL^2+\tfrac{1}{3}cL^3+\tfrac{1}{4}dL^4.$$

Where L is the length of the solid, *i. e.*, perpendicular distance between the planes of B′ and B″. The mean area is therefore

$$\Omega=V/L=a+\tfrac{1}{2}bL+\tfrac{1}{3}cL^2+\tfrac{1}{4}dL^3. \qquad (2).$$

But by (1)

if $l=0$; $\quad P=B'=a.$

if $l=\frac{1}{2}L$; $\quad P=M=a+\frac{1}{2}bL+\frac{1}{4}cL^2+\frac{1}{8}dL^3.$

if $l=L$; $\quad P=B''=a+bL+cL^2+dL^3.$

Therefore

$$B'+4M+B''=6a+3bL+2cL^2+\tfrac{3}{2}dL^3$$

and comparing with (2) we find

$$\Omega=\tfrac{1}{6}(B'+B''+4M).$$

The formula involving the base of the associate cone being of a less general character than that just written it is reasonable to expect of it more simplicity in application, as the sequel shows.

Consider for a moment a prismoid. Project it on a plane parallel to the bases. In this plane refer all points to any system of rectangular axes. Then if x', y' and x'', y'' be coördinates of a pair of corresponding corners in the bases. The coördinates of the corresponding corner in the mid-section will

be $$\tfrac{1}{2}(x'+x''); \qquad \tfrac{1}{2}(y'+y''),$$

while the coördinates of the corresponding corner of the base of the associate pyramid are

$$x'-x''; \qquad y'-y''.$$

The computation of the areas of the bases is the same in either case, while in order to compare the labor of computing the area of the mid-section with that required for the base of the pyramid, it is only necessary to see that in the respective coördinates we deal with sums in the one case and differences in the other, with the additional practical advantage always present that in the latter case the formula for mean area is in the shape of a correction applied to the average of end areas, the base of the pyramid in practical cases being small, whereas the mid-area generally exceeds the average of end areas.

While it is of no practical importance to the engineer it may nevertheless be interesting to apply the foregoing for the sake of illustration to the particular case of the railway prismoid.

The engineer in surveying his cross-section, refers the polygon to a system of rectangular axes in its plane and measures the coördinates of its corners with respect to the road bed line and the vertical through its center as axes of abscissa and ordinate respectively, calling them in order *distance-out* and *height*. Furthermore he takes no cognizance of the algebraic change of sign in coördinates, but merely calls them rights and lefts, cuts and fills respectively. The uniform method adapted for recording the field notes preserves the identily of the section.

Thus if b be half the road bed, h and m the height and distance-out to the right of center, k and n the corresponding measurements to the left, the record of the cross-section appears complete in the adapted form.

$$\frac{0}{b}\ \frac{k_s}{n_s}\ \cdots\ \frac{k_1}{n_1}\ \frac{d}{0}\ \frac{h_1}{m_1}\ \cdots\ \frac{h_s}{m_s}\ \frac{0}{b}$$

where d is the center height.

The area of any polygon in terms of the coördinates of its n corners being

$$2A=\begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix}+\begin{vmatrix} x_2 y_2 \\ x_3 y_3 \end{vmatrix}+\ -\ -\ -\ +\begin{vmatrix} x_n y_n \\ x_1 y_1 \end{vmatrix},$$

it is easy to see that the engineer's record of his cross-section is really a determinant for its double area. Thus the double area of the cross-section above recorded is

$$\frac{0}{b}\times\frac{k_s}{n_s}\times - - - \times\frac{k_1}{n_1}\times\frac{d}{0}\times\frac{h_1}{m_1}\times - - - \times\frac{h_s}{m_s}\times\frac{0}{b}.$$

In which the heavy lines join factors of positive product, dotted lines those of negative product. The sum of all the products is double the area.*

*This formula for the area of an irregular section was first given in Engineering News, Vol. XX, No. 39, in an article under the heading "A Cross-Section Mnemonic."

The cross-sections being written successively in the note book, each pair represents the bases of a prismoid whose lateral edges are noted, while on the ground by the engineer, by joining the coördinates of corresponding corners by a line in the notes.

Let the bases of a prismoid be

$$\begin{array}{cccccccc} \frac{0}{b} & \frac{K_s}{N_s} & & \frac{D}{0} & \frac{H_1}{M_1} & \frac{H_s}{M_s} & \frac{0}{b} \\ \\ \frac{0}{b} & \frac{k_s}{n_s} & \frac{k_1}{n_1} & \frac{d}{0} & \frac{h_1}{m_1} & & \frac{0}{b}. \end{array}$$

Coördinates in the same vertical are presumed to correspond without further indication.

The double mid-area for such a prismoid is to be computed from

$$\frac{0}{b}\ \frac{\frac{1}{2}(K_s+k_s)}{\frac{1}{2}(N_s+n_s)}\ \frac{\frac{1}{2}(K_s+k_1)}{\frac{1}{2}(N_s+n_1)}\ \frac{\frac{1}{2}(D+k_1)}{\frac{1}{2}n_1}\ \frac{\frac{1}{2}(D+d)}{0}\ \frac{\frac{1}{2}(H_1+d)}{\frac{1}{2}M_1}\ \frac{\frac{1}{2}(H_1+h_1)}{\frac{1}{2}(M_1+m_1)}\ \frac{\frac{1}{2}(H_s+h_1)}{\frac{1}{2}(M_s+m_1)}\ \frac{0}{b},$$

while that of the base of the associate pyramid comes from

$$\frac{K_s-k_s}{N_s-n_s}\ \frac{K_s-k_1}{N_s-n_1}\ \frac{D-k_1}{-n_1}\ \frac{D-d}{0}\ \frac{H_1-d}{M}\ \frac{H_1-h_1}{M_1-m_1}\ \frac{H_s-h_1}{M_s-m_1}.$$

Employing the same rule in either case as that given for the area of any ordinary cross-section, noticing that in the latter case the subtractions may change the sign of some of the products.

To apply the results to a numerical case, take the example in Henck's Field book, Art. 122, which he uses to compare methods.

$$\begin{array}{l} B'' = \frac{0}{9}\ \frac{4}{15}\ \quad \frac{8}{0}\ \frac{12}{27}\ \frac{0}{9} = 240 \\ \\ B' = \frac{0}{9}\ \frac{8}{21}\ \frac{13.6}{0}\ \frac{10}{24}\ \frac{0}{9} = 387 \end{array}$$

$$M = \frac{0}{9}\,\frac{6}{18}\,\frac{8}{10.5}\,\frac{10.8}{0}\,\frac{12.8}{13.5}\,\frac{11}{25.5}\,\frac{0}{9} = 355.55$$

$$B_p = \frac{4}{6}\,\frac{0}{21}\,\frac{5.6}{0}\,\frac{1.6}{-27}\,\frac{-2}{-3} = -88.2$$

Thus
$$\Omega = \tfrac{1}{6}(B'+B''+4M) = \tfrac{1}{2}(B'+B'') - \tfrac{1}{6}B_p$$
$$= 328.2\,.$$

The advantage in the computation lies so largely in favor of B_p as against M, that neglecting the advantage to be derived from the former as a correction to $\frac{1}{2}(B'+B'')$ it is preferable to use the simple method in cases where the actual volumes are to be computed.

Regularly and ordinarily in practice only three-level sections occur, and even then the computation of volume is further simplified by conceiving the surface ground to be determined by gauche quadrilaterals through each of which is passed a hyperbolic paraboloid, thus for each such quadrilateral we have one less corner in the mid-area and also in the base of the associate cone than would have occurred had the Henck prismoid been used instead. Evidently the introduction of the hyperbolic paraboloids does not interfere with the mid-area and the base of the associate pyramid remaining polygons, for in this surface one set of generators is parallel to the bases of the solid the other set for each surface moves always parallel to a fixed plane, therefore the corresponding element of the associate pyramid moves in a plane and traces a straight in the plane of the base. The reason for this simplification is not merely to save labor, but because in fact the volume for any gauche quadrilateral as determined by its hyperbolic paraboloid is exactly the arithmetical mean of the volumes which are determined by considering the diagonals of the quadrilateral successively as edges of a Henck prismoid. To prove this it is only necessary to prove the following:

Theorem:—The volume of the solid bounded by plane triangles, whose edges are the sides and diagonals of a gauche quadrilateral, is bisected by the hyperbolic paraboloid passing through the sides of the quadrilateral.

Let a and c be two opposite sides of the quadrilateral, the solid is composed of two pairs of triangles, a non-contiguous pair meeting in a, the other pair in c. Any plane parallel to a and c cuts the solid in a parallelogram, for it cuts the planes which meet in a in straights parallel to a, and those which meet in c in straights parallel to c. This parallelogram moving parallel to itself generates the solid. But the plane of this parallelogram always contains the element of the hyperbolic paraboloid of the (a, c) generation, and since this meets the other two sides of the quadrilateral it must be the diagonal of the parallelogram, dividing that figure into two equal triangles, each of which generates half of the solid.

Thus the ignoration of the diagonals, while in one particular prismoid may give an approximation to the earth volume not within the limit of error, in any series of consecutive prismoids must give a volume very near the true earth volume, since the cross-ridge and valley lines are just as likely to occur as one diagonal as the other.

It is under these assumptions then that engineers compute earth work volumes. Sections being reduced to three-level sections, the computation of mid-area and base of pyramid are correspondingly simplified. Thus in the regulation prismoid

$$\frac{\text{o}}{b}\ \frac{\text{K}}{\text{N}}\ \frac{\text{D}}{\text{o}}\ \frac{\text{H}}{\text{M}}\ \frac{\text{o}}{b}$$

$$\frac{\text{o}}{b}\ \frac{k}{n}\ \frac{d}{\text{o}}\ \frac{h}{m}\ \frac{\text{o}}{b}.$$

The mid-section becomes

$$\frac{\text{o}}{b}\ \frac{\frac{1}{2}(\text{K}+k)}{\frac{1}{2}(\text{N}+n)}\ \frac{\frac{1}{2}(\text{D}+d)}{\text{o}}\ \frac{\frac{1}{2}(\text{H}+h)}{\frac{1}{2}(\text{M}+m)}\ \frac{\text{o}}{b}.$$

The area of which can be computed as above, or may now be written out as a rule, thus

$$8M = 2b(\Sigma S) + (D+d)\,\Sigma O.$$

Eight times mid-area is $2b$ times the sum of side-heights $+(D+d)$ times the sum of the distance-out.

The consideration of the mid-area is useless and unnecessary for the base of the pyramid is

$$\frac{K-k}{N-n}\ \frac{D-d}{o}\ \frac{H-h}{M-m},$$

and its double area in algebraic form is

$$(D-d)\,(M+N-m-n)\,.$$

One-twelfth of this expression (which may be negative) subtracted from the average of end-areas gives the true mean area. It is upon this basis that the tables referred to have been computed. A formula may be written down at once for the correction to the average end-areas for any given cross-sections, but it would in general be too complicated for use.

An interesting point in connection with the cylindroid (prismoid) is the distance of its center of gravity from the plane of the mid-section, a value which is used in explaining the question of *long haul.* The formula for the running cross-section lends itself to an easy deduction of this.

Thus if X be the distance of the center of gravity from the plane of the base from which h is measured we have by the ordinary formula,

$$VX = \int_o^H Ph\,dh,$$

putting in the value of P in terms of h from above, and for V its value $H[\frac{1}{2}(B'+B'') - \frac{1}{6}B_c]$ we have, after integration,

$$H[\tfrac{1}{2}(B'+B'') - \tfrac{1}{6}B_c]\,X = H^2\,[\tfrac{1}{6}B' + \tfrac{1}{3}B'' - \tfrac{1}{12}B_c].$$

The distance of the center of gravity from the mid-section is

$$x = X - \tfrac{1}{2}H.$$

Substituting in the above we get

$$x = \frac{H}{6}\,\frac{B'' - B'}{B'' + B' - \frac{1}{3}B_c}.$$

Whence the approximate formula used by the engineer

$$x = \frac{H}{6}\,\frac{B'' - B'}{B'' + B'}.$$

Since $\frac{1}{3}B_c$ is practically small when compared with $B'' + B'$.

THE BEGINNING OF MATHEMATICS.

By Prof. W. B. RICHARDS, Rolla, Mo.

II GEOMETRY.

—*μὴ εἶναι βασιλικὴν ἄτραπον ἐπὶ γεωμετρίαν.*

Euclid. ap. Procl. Diadoch.

It is unfortunate that the adjective—Greek—in prevailing use to designate whatever pertains to the people of whose intellectual labors we are about to speak, should suggest but a part—and what at some periods was a minor part—of the territory through which this people was scattered, and in which their mental activity had its first field. The Hellenes themselves knew their country, as their descendants still know it, only as Hellas. The name Greece, given to it by the Romans and adopted by common consent of later times, is derived from *Γραικοί*, the name of a small tribe with whom the Romans first came in contact. It is not mentioned by any writer earlier than Aristotle. The employment of the term Greek, while sanctioned by usage, is apt to be misleading to one better acquainted with modern geography than with ancient history. The national spirit was in a largely measure wanting among the Hellenes, and the ties which they recognized were based upon ethnic, rather than geographic relations. The name Hellas was broad enough to include not merely the dwellers in the little peninsula which now bears the name of

Greece, but all, under whatever sky, who might claim descent from them—not merely the walker in the groves at Athens, the hardy Spartan, or the Messenian mountaineer, but him as well of the same stock who had his household gods by Leucadian steep, or breathed the soft air of Italian Sybaris. Indeed the first fruitage of what was to be so glorious a springtime sprang upon other soil than that which lies between Olympus and Taenarum. It was among the colonies, not on the main-land, that were made the first steps in Literature, in Philosophy and in Science. Except the Boeotians, Hesiod and Pindar, no poet of the first rank (if the former's theogony and rough and ready expressions of practical wisdom entitle him to be named as an exception) acknowledged as his home what we know as Greece until the time of Aeschylus. Homer, if we admit that there was one such man who wrote the poems attributed to him, was an Ionic Greek, living in Asia Minor or on one of the islands that fringe its coast. The biting iambics of Archilochus, the noble lyrics of Simonides, Anacreon's praise of Love and Wine, the mutual sighs of Alcaeus and soft, slandered Sappho, all come from lips that learned to lisp numbers amid the Aegean isles. All the early philosophers from Thales to Sacrates, that is for two centuries, were natives of the one or the other of the Hellenic colonies.

The Ionic Greeks, who had their seats along the middle part of the western coast of Asia Minor and in the islands adjácent to it—"Sons of Javan", as the Scripture calls them—were the earliest movers in the work of Greek culture. Their characteristics as a people and their situation combined to give them this precedence. The Ionians were distinguished among the other Greeks for their quickness, their viracity, their readiness to receive impressions. The stuff of which they were made was far more fictile than that of their Aeolic or Doric kinsfolk. They possessed in the highest degree among Greeks the qualities that distinguished the Greeks from their contemporaries.

Their location, in the direct track of the western process of civilization, and their commercial relations with the Egyptians and the Phoenicians, contributed to make them first among Aryans to feel the impetus toward scientific investigation which an acquaintance with the attainments of these people would give.

The meagreness of the contemporary historical records does not enable us to speak with definiteness and certainty as to the exact connection between the incipient Greek culture and the achievements of its predecessors. Ueberweg (Hist. of Philosophy, Vol. I., p. 31,) says: "To what extent the philosophy of this age (and hence the genesis of Greek philosophy in general) was affected by Oriental influences, is a problem whose definite solution can only be anticipated as the result of the further progress of Oriental and, especially, of Egyptological investigations. It is certain, however, that the Greeks did not meet with fully developed and completed philosophical systems among the Orientals," The same general fact is true of Science. Nor are the traditions of either the Greeks or the Orientals entirely trustworthy. It would not be strange if the early Greeks, anxious to lend a flavour of antiquity to their teachings, should have attributed their origin to the Egyptians, nor if the national pride of this latter people first consented to the attribution, and then insisted upon it, until they, and the world at large, placed far more stress upon the indebtedness of the younger to the older people than is justified by the facts. The work of the Orientals is not to be neglected in estimating the influences that brought about the beginnings of Science, yet on the other hand we need to guard against the danger of ascribing to it a part in the history of Science in general, and Mathematics in particular, beyond that which it really played. What they did was a leading up to Science rather than a beginning of it, and the debt due to them from the Greeks and all later nations was not comparably so much for actual contribution as it was for suggestion and incentive.

The beginning of Science is signalized by the appearance for the first time of a single name in connection with the advancement of knowledge. The Assyrians had an Astronomy, with copious records of observations made, but no astronomer; the Egyptians had an inkling of Geometry, but no geometer. Some progress in learning may be made under the push of natural laws by a people, working without concert, yet happening the one occasionally to cap the discovery of another with a greater; but no body of thought assumes the proportions of a science until its scattered fragments have been collected and fused together in the crucible of a single brain.

The same venerable personage stands at the head of the long list of philosophers, astronomers and mathematicians. Indeed at this early period to be one of these was well nigh being the others.

Thales of Miletus was born in the Ionic city of that name on the western coast of Asia Minor about the year 640 B. C. Herodotus (Book I., c. 170) says he was of Phoenician descent. Diogenes Laertius gives Plato as authority for the tradition that his ancestry might be traced back to Cadmus, who first introduced letters into Greece, and Zeller agrees with this view. Another account makes him out a native Milesian of pure Greek blood. Whether or not his family relations were such as would involve a connection with the people by whom his compatriots were being imbued with learning, the circumstances of his birth placed him in the immediate path of the westward flowing stream of knowledge. Thales, who enjoyed among the Greeks a reputa-

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*Lives of the Philosophers, I., 1.

tion for wisdom which we know not whether to attribute more to his own worth or to the obscurity of the period, was by common consent acknowledged first of the seven sages. His most valid claim to the admiration of his contemporaries and posterity as well, was the alleged prediction of an eclipse of the sun, which took place in 585 B. C. This was the same obscuration otherwise famous in history as having occurred just as the Lydian and Median Kings were about to join battle on the banks of the river Halys, and which so filled these barbarous potentates with awe that they at once declared a mutual peace. Herodotus, who was too fond of a good story to be embarrassed by considerations as to its truth, recites this legend (B'k. I, c. 74), and later authorities follow him. There are, however, we regret to say, serious, if not insurmountable, difficulties in the way of our lending credence to this pleasant narrative. The state of Astronomical science at the time would scarcely warrant the belief in the possibility of so exact a calculation. It is not at all unlikely that Thales was acquainted with the Assyrian "Saros," a period of eclipses covering 233 lunations, or even the longer period of 600 years. But while eclipses of the moon were predicted by means of these cycles it is disputed whether eclipses of the sun could be foretold in the same way. To have predicted this phenomenon for any definite locality moreover, would have involved a knowledge of the sphericity of the earth, which was sufficiently foreign to Thales' conceptions. Plutarch says that Thales was engaged in commerce, and all the authorities agree that in this way he was led to Egypt, and became acquainted with the Egyptian rudiments of Geometry. Diogenes Laertius quotes Hieronymus of Rhodes as asserting that he never had any teacher save when he went to Egypt and associated with the priests.*

*Lives of Philosophers, I., 6.

Hence he obtained the inspiration of his Geometric investigations; his speculative spirit seized upon the crude fragments of truth tortured from centuries of experience and observation, and began to build beyond them into the Empyrean. Proclus (Ad. Euclid. p. 19) says:

*"Θαλῆς δὲ πρῶτον εἶς Αἴγυπτον ἐλθὼν μετήγαγεν εἶς τὴν Ἑλλάδα τὴν Θεωρίαν ταύτην καὶ πολλὰ μὲν αὐτὸς εὗρε, πολλῶν δὲ τὰς ἀρχὰς τοῖς μετ᾽ αὐτὸν ὑφηγήσατο, τοῖς μεν καθολικώτερον ἐπιβάλλων, τοῖς δέ αἰσθητικώτερον."**

The following propositions are attributed to Thales:

(1.) The circle is halved by its diameter.

(2.) The angles at the base of an isosceles triangle are equal.

(3.) The vertical angles formed by the intersection of two right lines are equal.

(4.) Two triangles are equal when they have one side and two angles of the one equal respectively to the corresponding parts of another.

(5.) The angle inscribed in a semi-circle is a right angle.

Diogenes Laertius says, "Pamphile relates that he (Thales), having learnt Geometry from the Egyptians, was the first person to describe a right angled triangle in a circle, and that he sacrificed an ox in honor of his discovery. But others, among whom is Apollodorus, the calculator, say that it was Pythagoras who made this discovery. It was Thales also who carried to their greatest point of advancement the discoveries which Callimachus in his iambics says were first made by Euphebus the Phrygian, such as those of the scalene angle and of the triangle, and of other things which relate to investigations about lines."**

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*Thales, after having journeyed into Egypt, brought back this science (Geometry) to Greece and both discovered many things himself and handed down to his successors the elements of many things, approaching some in a more general manner, some in a more experimental.

**Lives of Philosophers, I., 25.

(6.) The homologons sides of similar triangles are in proportion. Plutarch distinctly ascribes this to him.

We are confronted with the difficulty of which we have spoken in attempting to determine what part of the enunciations accredited to Thales was derived from his intercourse with the Egyptian priests, and what was original with him. We can readily see how the conclusions, (1), (2) and (3), could be reached inductively from observation of particular cases, and might belong to that portion of his teachings at which Proclus says he arrived *αἰσθητικώτερον*—in a more sensible (empiric) manner—a portion which may fairly be assumed to stand for his immediate acquisition from the Egyptians; while (4), (5) and (6) would seem to belong to that part proved *καθολικώτερον* —more generally—and to be the product of his own invention.

The proof of (5) [Euclid I, 31,] involves (2) and the principle that the sum of the angles of any triangle is equal to two right angles. This would demand that Thales should be acquainted with the last named proposition—that is if a general proof of (5) was offered. Proclus asserts that the theorem concerning the angles of a triangle's being equal to two right angles was first proved in a general way by the Pythagoreans, but it was probably known to early mathematicians as a fact of observation.

Two applications of this new instrument, Geometry just being fitted to the worker's hand, to the solution of practical problems—marvelous enough they must have seemed to the ancients—are handed down as having been made by Thales. These were the determination

(1) of the distance of a ship at sea;

(2) of the height of the pyramids by their shadows.

These problems are interesting, besides in other respects, as showing the influence of environment in determining the direction of mental effort, and confirming the principle, upon which we touched in the preceding paper, that inventions in the field of science spring from the suggestion of practical questions. The residence of Thales upon the coast and among a maritime people, naturally presented the first problem to his inquiring mind, and furnished an incentive to the solution of it which would have been absent had he spent his days inland; while his travels in Egypt, beneath the shadow of the lofty pyramids, could not fail to stir his spirit up to an attempt to compass what seemed the impossible feat of measuring those inaccessible heights. And it is more natural to suppose that the important general theorem that the sides of equiangular triangles are proportional, which it is generally assumed that the solution of these problems presupposed, was discovered in the attempt to solve them, than that it occurred to Thales in a purely abstract way, and that the questions were afterwards resolved by its aid.

Diogenes Laertius quotes Hieronymus of Rhodes as saying that "He measured the pyramids, watching their shadow and calculating when they were of the same size as that was." Others give an account of the feat which would involve the use of Theorem (6) alone. Obviously enough both of these problems might be solved without using (6), by means of (2).

So much stands accepted in history as the tangible work of Thales. But remarkable as were these achievements in comparison with aught that had been done before, they in themselves mark but a fraction of the service of Thales to later science. The prime element of this contribution was the institution of a method of inquiry which passed beyond the individual instance, and sought to lay an unchanging foundation in the principles of abstract truth. Science was set on foot and might run its victorious course.

The immediate successors of Thales in the Ionic school of Philosophy, while they took up the physical speculation in which he had set an example, do not seem, so far as any records show, to have made any advancement in Mathematical invention. The next step in the development of Geometry was the work of the Italic school—to accept the ancient geographical division—that is, of Pythagoras and his followers. The name of Pythagoras is one suggestive at once of myth and mystery. The real figure of the man has come down through history girt about with an increasing nimbus of mystic tradition. He is the hero of a cycle of philosophic legend. Not only his own age but later ages have ascribed to him the possession of supernatural power, and have painted him and his school in the colors of romance. Out of the mass of tradition which has been handed down to us, it is impossible to assert confidently how much is fact and how much fiction. His life, if the reports of the ancient historians are accepted, was as varied as the range of his speculations.

He was the son of Mnesarchus, the Tyrian, a seal-engraver, and was born in the island of Samos about 582 B. C. He studied at Lesbons under Pherecydes and at Miletus under Thales and Anaximander. He visited Egypt and spent twenty-seven years at Memphis and Thebes communing with the priests and sages. When Cambyses over-ran Egypt in 525 B. C., Pythagoras was among the captives, and was carried off to Babylon, where he was held as a slave for some years. Here he became acquainted with the learning of the Chaldaeans, and gained as well an introduction to the religion of the Hindus. Having obtained his liberty, he visited in turn Crete, Sparta, Elis, and Delphi, and returned to his native isle to establish a school. Discontented with the tyranny of Polycrates, he proceeded to Italy and founded at Crotona "in the house of Milo", a school which soon attracted a large number of attendants.* Interference in local politics caused his own

*Diogenes Laertius says three hundred.

banishment, the burning of his house and the dispersion of his followers. He retired to Metapontum where he died soon after, (about 490 B. C.)

Such the narrative that might be constructed by one liberally credent. The statements as to his travels and adventures in the East, it has been observed, may either be statements of fact or an invention to connect his teachings with earlier originals. There are reasons to believe that he did travel in Egypt; the rest is more doubtful. Cicero says, (De. Fin., V. 29, 87) "*Aegyptum lustravit.*" He also says that Pythagoras came to Crotona in Ol. 62. 4 (529 B. C.,) (*Rep*. II. 15). If this be true we should have to give up the story of his Babylonish captivity. We must ever regret that the history of Pythagoras' life, written by Theano, a beautiful young girl whom he espoused in his seventieth year, has not come down to us.

The society which Pythagoras founded was on its scientific side a school of philosophic inquiry and instruction, while on its ethic side it partook of the nature of a religious brotherhood. The search for truth was combined with a rigid personal discipline. Intending disciples were said to have been subjected to a long period of probation, of which strict obedience and absolute silence were the cardinal features. Diogenes Laertius says that this period was five years. The use of animal food was permitted only within certain restrictions. Certain vegetables were tabooed, and celibacy was inculcated. Thus we see that the ancient Pythagoras and the modern Tolstoi are alike as regards both theory and practice.

The speculations of the school took a wide range—over philosophy, astronomy, mathematics, music. Best known, perhaps, of their tenets is that of the transmigration of souls, which their great founder is thought to have imbibed in his Oriental wanderings. He, himself, claimed to be a son of Mercury, and to have existed in many previous shapes. He said that Mercury offered him any gift save immortality and that accordingly he requested that, whether living or dead, he might preserve the memory of what had happened to him. So was his existence continuous.*

The philosophical conceptions of the Pythagoreans were strangely blended with arithmetic considerations. The whole system of the universe was held to depend upon the relations between numbers. "The Pythagoreans seem" says Aristotle, "to have looked upon number as the principle and, so to speak, the matter of which existences consist." "Number," says Philolaus, (the successor of Pythagoras), "is great and perfect and omnipotent, and the principle and guide of divine and human life."†

Proclus states, in his commentary on Euclid's elements, that the word "mathematics" originated with the Pythagoreans. The same author says that the Pythagoreans made a four-fold division of mathematical science, its parts corresponding to Arithmetic, Music, Geometry and Astronomy, respectively.‡ Diogenes Laertius relates that "It was Pythagoras also who carried Geometry to perfection, after Moeris had first found out the principles of the elements of that science as Aristiclides tells us in the second book of his History of Alexander."§ The Pythagoreans defined a point as "Unity having position." They showed that the plane around a point is completely filled by six equilateral triangles, four squares, or three regular hexagons. (Proclus). Eudemus attributes to them the theorem that the interior angles of a triangle are equal to two right angles, and gives their proof, which is substantially the same as that given by Euclid. We have it stated upon the same authority (quoted by Proclus in his commentary) that the problems relating to the application of areas, the construction of

*Diogenes Laertius. Lives of Philosophers, B'k. VIII, c. 4.

†Encyclopædia Britannica, Vol. XX. p. 144.

‡Encyclopædia Britannica, Art. Pythagoras, p. 146.

§Diogenes Laertius, Lives of Philosophers, VIII, 11.

the five regular solids, and the discovery of irrational quantities were all due to Pythagoras. Three of the five regular solids, the tetrahedron, the cube and the octahedron, were known to the Egyptians, and occur in their architecture. Pythagoras discovered the other two, the dodecahedron and icosahedron, and shewed how to construct them all. The discovery of the existence of irrational quantities was one of the most notable made by the Pythagorean school; it may have arisen from an attempt to express the length of the diagonal of a square in terms of a side. It paved the way for the general treatment of proportion found in Euclid—a treatment which holds as well for incommensurable as for commensurable magnitudes.

The theorem best known in connection with the name of Pythagoras—in fact frequently cited as the Pythagorean theorem—is that which asserts that "the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides." This proposition is distinctly attributed to Pythagoras by Vitruvius, Diogenes Laertius, Proclus and Plutarch. Diogenes Laertius says, "And Apollodorus, the logician, records of him that he sacrificed a hecatomb when he had discovered that the square on the hypotenuse of a right-angled triangle is equal to the squares of the sides containing the right angle. And there is an epigram which is couched in the following terms:

> "When the great Samian sage his noble problem found,
> A hundred oxen dyed with their life blood the ground."

This, it will be observed, is just one hundred times the amount of gore which the same author represents Thales as having spilled after having inscribed a right angle in a circle. Plutarch in his work on "Isis and Osiris," dealing with the mysteries of Egyptian religion and learning, asserts that the ancient Egyptians knew that a triangle whose sides contain three, four and five parts respectively is right-angled, and that the square

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*Lives of Philosophers, VIII, 11. (Yonge's Translation.)

on the longer side is equal to the sum of the squares on the other two. It has been very plausibly suggested, too, that the fact might have become known to them in a special case by the observation of the tesselated floors common in Egyptian buildings. The square on the diagonal of one of the square tiles would be seen to contain four of the isosceles right triangles into which that diagonal divided the block, while each of the squares on the sides contained two such equal triangles.* To Pythagoras, however, belongs the merit of having given the first general proof of the proposition—a proof the same, there is no reason to doubt, as that now given in our Geometries. Many different proofs of this historically interesting proposition, it may be noticed in passing, have been offered; a number of them have been collected in one publication by a German named Hoffman.

Materials do not exist for writing a succinct history of the progress of Mathematics in the century and a half between Pythagoras and Euclid. We know that it was a period in which there was much geometric study, and in which considerable additions were made to the fund of Mathematical knowledge, but of these only meagre and fragmentary records remain. There were three problems which especially engaged the attention of mathematicians:

1. The duplication of the cube.
2. The trisection of an angle.
3. The quadrature of the circle.

—two, three, four, one might say as a mnemonic. The work of some few men—leading like stepping-stones from the one to the other of the great names mentioned—deserves to be noticed, Our account is in the main taken from Marie's "Histoire des Sciences Mathematiques et Physiques." Hippocrates of Chios, (bn. 450 B. C.) wrote a book on the elements of

*See the article--Pythagoras--in the Encyclopædia Britannica, from which many of the above statements are derived.

Geometry; he was the first to reduce the problem of the duplication of the cube to that of the insertion of two mean proportionals between two given magnitudes. This Hippocrates is not to be confused with the physician of the same name.

Zenodorus (bn. 450 B. C.) is the author of the oldest work on Geometry which has come down to us. This work was preserved by Theon, of Alexandria, in his commentary on the "Syntax" of Ptolemy. Zenodorus attempted to combat the opinion, then common, that equal contours enclose equal surfaces.

Archytas (440—380 B. C.), according to Diogenes Laertius, "was the first person who applied mathematical principles to mechanics and reduced them to a system. and the first also who gave a methodical impulse to descriptive geometry in seeking in the sections of a demicylinder for a proportional mean which should enable him to find the double of a given cube." He was also the first person who ever gave the geometrical measure of a cube, as Plato mentions in his Republic.

The great Plato (430 B. C.—347 B. C.) left no work on Geometry, but he rendered mathematics a signal service in directing the attention of his disciples to the study of the conic sections, and in the invention of loci for the solution of the problems mentioned above.

Eudoxus of Cnidus (409 B. C.—356 B. C.) was a man of varied learning. Apollodorus in his "Chronicles" says that "he was the inventor of the theory of crooked lines."* Archimedes, in his letter conveying to Dositheus, his treatise "On the Sphere and Cylinder," attributes to Eudoxus the theorems: "a pyramid is the third part of a prism having the same base and altitude; a cone is the third part of a cylinder having the same base and altitude." Eudoxus gave a solution of problem (1) which Eratusthenes regarded as excellent, but which is lost.

*Diogenes Laertius, Sub. nomine.

Menaechmus (bn. circa. 395 B. C.) studied particularly the elementary theory of conics. He gave a solution of problem (2) worthy to be mentioned. Using the modern notation it is as follows: Let two parabolas have their axes at right angles, and let p and q be their respective parameters. Then the equations will be $y^2=px$ (1) and $x^2=qy$ (2)

$$\text{from (1) } p:y=y:x$$

$$\text{and from (2) } q:x=x:y;$$

for the point of intersection,

$$p:y=y:x=x:q.$$

So to find the required proportionals between two quantities, with these lines as parameters construct two parabolas with axes at right angles; then the coördinates of the point of intersection will be the required proportionals.

The name of Euclid has become literally synonymous with Elementary Geometry. Of no man can it be more truly said "He lives in his work." Indeed he scarcely lives for us outside of it. His fate is that of some of the world's greatest—to have handed down an utterly imperishable work, and yet to have left upon history but scant impress, if any, of his own personality—the fate, for instance, of Homer and of Shakspere. Euclid's life, falling midway between the age of fable and that of careful and minute historic record, lacks the wealth of legend and tradition with which the stories of Thales and of Pythagoras were richly woven about, and fails of the full narration which it might have received in later years. Nothing is definitely known as to his parentage and place of birth. He flourished in the first half of the third century B. C. Proclus asserts that he was younger than the associates of Plato, but older than Eratosthenes (276—190 B. C.) and Archimedes (287—212 B. C.) The new Egyptian city, Alexandria, sheltering the ashes and perpetuating the name of its great founder, was just rising into importance as a centre of culture.

Ptolemy Soter, to whom Egypt had fallen in the division of Alexander's spoils, had founded the great Alexandrian library, and gathered about him a circle of *savants*. Among others Euclid was invited thither, and here founded his school of mathematics. The King himself—so the story goes—was led by his enthusiasm for learning to become a pupil of the great mathematician, but finding the "Elements" rather more difficult reading that that to which his kingly patience was accustomed, inquired if there was not some easier way of learning the subject. To which Euclid made the celebrated reply which stands at the head of this paper: "There is no royal road to Geometry." Two of Euclid's mathematical works have been preserved, the Elements (Στοιχεῖα) and the Data (Δεδομένα). Euclid's Elements have been accepted in all later times as embodying the essential requirements of primary geometrical teaching. Boetius, senator and philosopher, the last of the Romans of the old school, is said to have translated a part of the Elements into Latin (6th century), but in the lack of consistency among the manuscripts, critics are inclined to doubt their authenticity. The Arabs, to whose labors we are so largely indebted for the preservation of learning during the Dark Ages, busied themselves with translations of Euclid; one such translation by Nasr-ed-Din Ibn-Hassan, the Persian astronomer of the thirteenth century, appeared at Rome in 1594. The first printed edition was a translation from the Arabic by the Italian, Campano, which was made in 1482. About twenty years later a translation from the Greek was made by Zamberti, and printed at Venice. Our one English edition containing all the works of Euclid is the Oxford edition, published by Dr. David Gregory in 1703, with the title *Εὐκλείδου τὰ σωζόμενα*. The compilation which has formed the basis of later English works on the subject, is the one given forth in 1756 by Dr. Robert Simson, Professor of Mathematics in the University of Glasgow. It comprised the first six books of Euclid, some of the eleventh book and two propositions of the twelfth. A favorite text is that of the late Prof. Todhunter, which is founded on Simson's. On the continent Euclid has not been so strictly followed as in Great Britain. Our American treatises on Elementary Geometry are generally modelled after the French, a large number of them being merely adaptations of the work of the amiable and great Legendre.

The "Elements" of Euclid consisted of thirteen books, to which were added two others, on the five regular polyedra, of which the Alexandrian, Hypsicles, is supposed to have been the author. Euclid was both a collector and an originator. No means exist of discriminating exactly the two parts of his work. We have seen that certain propositions had been discovered before him. How many more of those which are gathered in his collection were due to others we cannot say. "Euclid," says Proclus, "put in order many things discovered by Eudoxus, perfected what Theaetaetus had begun, and demonstrated more rigorously what had previously been too loosely proved."

The first book begins with the definitions, the postulates, and the axioms. Here Euclid is laying the foundation of his science, and just here does he meet with the largest amount of cavil on the part of critics. There is certainly ground for objection to some of his statements, but a discussion of them would be out of place here. A domain far beyond the ken of Euclid's restrictions has been glimpsed after two thousand years by the inventions of Hamilton, Grassman, and others.

The first proposition is the problem to describe an equilateral triangle on any sttaight line as a side, and the first theorem is as to the congruency of two triangles which have two sides and the included angle of the one equal to ths corresponding parts of the other. The book ends with the Pythagorean theorem and its converse. The second book treats of the relations between squares and rectangles formed on certain lines and their segments. It contains two problems—to divide a line into extreme and mean ratio, and to describe a square equal to a given rectilinear figure. Stated algebraically, the first problem is to find x so that $a\ \ (a-x)\times x^2$ or $x^2+ax=a^2$, and so it involved the solution of one form of a quadratic equation. The third book is on the circle. The fourth book consists entirely of problems on the inscription and circumscription of circles and polygons, including the problem to construct an isosceles triangla, having each angle at the base double of the angle at tho vertex, which is used in inscribing a regular pentagon in a circle. The fifth book is devoted to Euclid's celebrated treatment of proportion. The essence of the treatment lies in the definition of proportionality; and its superiority consists in the generality which flows from this definition, and renders the method applicable to incommensurable magnitudes as well as to commensurable. The sixth book contains a number of theorems and problems involving the application of proportion. So far the enunciations are all for figures in one plane. Books seven, eight and nine are on Arithmetic. Book ten is on incommensurables. Books eleven, twelve and thirteen are chiefly on Solid Geometry.

The Data of Euclid comprised, according to Pappus, ninety propositions; in the extant editions ninety-five propositions are included under the designation. Dr. Simson has left an edition of these also. The Data were propositions in which it is required to prove that certain things being given certain others may be determined—that is are *potentially* given, since involved in the hypothesis. The work was intended as a kind of supplement or appendix to the elements, designed to facilitate the application of the principles contained in them to the solution of problems.

As examples, we may cite:

"If from a given point a line is drawn, touching a circle given in position, the line is given in position and magnitude."

And (Prop. 6), "If two quantities are to each other in a given ratio, the quantities compounded of the two shall be to each other in a given ratio."

Among the lost works of Euclid of which we have record are: Two books on Plane Loci, four on Conics, and three on Porisms. Simson thought that the books on Plane Loci treated of curves of double curvature, an opinion which was shared by the historian, Montucla. Chasles, in the introductory lecture delivered upon the inauguration of his course in Higher Geometry at Paris, took the view that they treated of surfaces of revolution of the second degree and the sections of them by planes—with whom M. Marie, in his recent history of Mathematics, agrees.

Pappus says that Euclid wrote four books on Conics which formed the basis of the great work of Apollonius, the "Sublime Geometer." Apollonius in his letter transmitting his treatise to Eudemus, says that in his first four books he had elaborated that which had been done before him, and especially mentions a certain problem which had been solved by Euclid only in a special case. We have, however, no information which enables us to speak with any degree of certainty of the content of Euclid's work.

The Porisms of Euclid present one of the profoundest of mathematical enigmas. What did Euclid mean by a Porism, and what were the propositions which he enunciated under that name? Commentators and editors, among them some of the brightest of geometers, have essayed the solution of this question. Albert Girard, in the first half of the seventeenth century expressed the hope that he might restore the lost Porisms, and Fermat, a little later, touched upon the same subject. In 1776 appeared a posthumous work of Simson's, "*De Porismatibus tractatus; quo doctrinam Porismatum satis explicatam et in posterum ab oblivione tutam fore sperat Auctor.*" In our own century the great Chasles has made a brilliant effort at the re-establishment of the lost work in his "*Trois Livres de Porismes d' Euclide retablis, etc.*"

Our definite information concerning Porisms is found in a notice given by Pappus in the seventh book of his "Mathematical Collection," and a brief mention by Proclus in his commentary on the first book of the elements. Pappus says that the Porisms of Euclid were an ingenious collection of a number of propositions, serving for the solution of the most difficult problems; that the ancients defined Theorem, Problem and Porism as propositions respectively in which it is required to prove, to construct, and to find something. Proclus gives a similar definition of Porisms which, he says, occupy a place intermediate between theorems and problems. Simson defined a porism as "a proposition in which it is required to show that one thing is given, or several things are given, which, as well as any one of an infinite number of other things not given but of which each one bears the same relation to the given things, have a certain common property described in the proposition." Playfair, professor of Mathematics in the University of Edinburgh, in a memoir suggested by Simson's work, defines porism as "a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of an infinite number of solutions." Chasles, after noticing the other attempts at the definition of a porism, presents this one: "Porisms are incomplete theorems expressing certain relations between variables following a common law." Pappus states that there are thirty-eight Lemmas for the three books of Porisms, from which are deduced one hundred and seventy-one theorems.

We believe that this perplexed problem has been still more obscured by the attempts at its elucidation. We believe that those who have attempted to give a definition of "Porism" have been groping around in the dark for what was not there. The name by which Euclid designated the propositions in question

—πόρισμα (gain, extra acquisition)—is the same as that applied to the immediate deductions from theorems which we translate. "Corollary" There is no notice of any definition of Porism given by Euclid; that which Pappus quotes, he attributes to "the ancients." We believe that Euclid in calling these propositions Porisms was not intending to distinguish any essential characteristic resident in all the enunciations, but simply labeling them as, like the Corollaries, "additional propositions"—a "gain" from previously deduced theorems. We do not think that Euclid intended to apply the name Porism to a class of propositions distinct, in some mysterious and hitherto inexplicable way, from both propositions and theorems. No rational explanation of Porisms has ever been offered which did not include them under the one or the other. Pappus, in his notice of them, quoted above, calls them "theorems." Simson says, "A Porism is a proposition in which it is required to *demonstrate*, etc.," and this, according to the definition in Euclid's Elements, certainly constitutes a theorem. Chasles, we have just seen, defines them as "incomplete theorems." The diversity of expression among geometers who have discussed Porisms is due to an effort to frame a definition which shall comply with Pappus' representation of them as different in some way from both theorems and problems, and shall be comprehensive enough to include under it all the cases in question. The probability that Euclid used the word Porism in the sense which we have suggested is increased by the consideration that Diophantus gave the same title to a treatise of his having no connection with geometry, and to which accordingly the definitions of Porisms ordinarily given could not apply. A writer on the subject, speaking of the work just cited, says: "These propositions are not, however, all similar in form, and we cannot by means of them grasp what Diophantus understood to be the nature of a porism." Is is not probable that these were simply additional propositions suggested by the line of thought contained in his great Arithmetic, and that he meant no more in calling them porisms?

In subject matter the Porisms were what Euclid might have called a "Modern Higher Geometry;" they contained, among other things, the elements of the theory of transversals and some properties relative to the anharmonic ratio of four points, and thus were an anticipation of what is known to-day as Modern Higher Geometry.

The following are some of the authorities for the period treated of:

Diogenes Laertius' Lives of Philosophers--written in second century. Uncritical in spirit and unscientific in method, but interesting in matter.

Pappus. Lived at Alexandria in fourth century. His "Collection" in eight books gives results attained by previous mathematicians, together with original discoveries. Very important.

Proclus. The Neo-Platonist, bn. at Constantinople, 412 A. D. Studied at Alexandria. Wrote a commentary on first book of Euclid's Elements.

Ueberweg's History of Philosoph. §§12, 16.

Grote's History of Greece, Vol. II., c. 37.

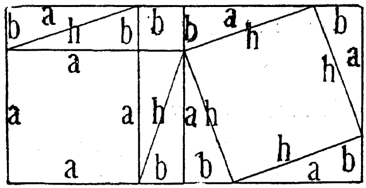
Marie's Histoire des Sciences Mathematiques et Physiques, Vol. I. Recently completed in twelve volumes. (Gauthier-Villars, Paris.)

Chaslés' Les trois Livres de Porismes d' Euclide re'tablis. (Mallet-Bachelier, Paris.)

Encyclopædia Britannica, articles, Thales, Pythagoras, Geometry, Porism.

A NEW ELEMENTARY DEMONSTRATION OF THE PYTHAGOREAN THEOREM.

By Dr. WILLIAM B. SMITH, Columbia, Mo.



From each of two congruent squares cut away four congruent right triangles; of the one there is left the square on the hypoteneuse of the right triangle; of the other, two squares on the legs of the right triangle; hence the truth of the proposition.*

*[The above demonstration of the Pons Asinorum is so good and simple that it is difficult to believe it new. We are inclined to think it is, if for no other reason than that Todhunter in his Edition of Euclid's Elements in remarking on the Theorem in the notes, gives there as the most interesting of the many demonstrations one in which any two unequal squares are used and the proof is not so good as the above.

The largest collection of demonstrations of this proposition seems to be a dissertation by Joh. Jos. Ign. Hoffmann, entitled "Der Pythagorische Lehrsatz . . . Zweyte . . . Ausgabe. Mainz, 1821. This we have not been able to examine.--(Eds.)]

SOLUTIONS OF EXERCISES

1.

Two vertices A and B of a triangle A B C describe straight lines which meet at the angle ω; show that the area of the curve described in their plane by the vertex C is

$$\Omega = \tfrac{1}{2}\pi\,(a^2+b^2+c^2-4\triangle \text{ ct } \omega).$$

$\triangle$ being the area of the triangle A B C. [*W. H. Echols.*]

SOLUTION.

Let the paths of A and B meet in I.

In any position of A B draw the cir-cum circle A B I centered at O whose radius is r. Put C O$=\delta$.

Then the path of C is an ellipse whose semi-axes are $\delta+r$ and $\delta-r$. (Sc. B., Vol. I., No. I.)

Join O A and OB, let O A B$=\alpha$.

Then $$\alpha+\omega=\tfrac{1}{2}\pi.$$

The triangle O C A gives

$$\begin{aligned}\delta^2 &= r^2+b^2-2rb \text{ co}\,(\text{A}+\alpha),\\ &= r^2+b^2-2rb \text{ co}\,(\tfrac{1}{2}\pi+\text{A}-\omega),\\ &= r^2+b^2-2rb \text{ si}\,(\text{A}-\omega).\end{aligned}$$

The triangle O C B gives in like manner,

$$\delta^2 = r^2+a^2-2ra \text{ si}\,(\text{B}-\omega).$$

Whence results

$$\begin{aligned}2(\delta^2-r^2) &= a^2+b^2-2r[b \text{ si}\,(\text{A}-\omega)+a \text{ si}\,(\text{B}-\omega)],\\ &= a^2+b^2\\ &\quad -2r[\text{co}\,\omega\,(b \text{ si A}+a \text{ si B})-\text{si}\,\omega\,(b \text{ co A}+a \text{ co B})],\\ &= a^2+b^2-\frac{\text{c}}{\text{si}\,\omega}[2h \text{ co}\,\omega-\text{c si}\,\omega].\end{aligned}$$

Since $c = 2r \text{ si } \omega$, $b \text{ co } A + a \text{ co } B = c$, $b \text{ si } A = a \text{ si } B = h$.

Hence
$$\Omega = \pi(\delta + r)(\delta - r) = \pi(\delta^2 - r^2),$$
$$= \tfrac{1}{2}\pi \left\{ a^2 + b^2 - \frac{c}{\text{si } \omega}[2h \text{ co } \omega - c \text{ si } \omega] \right\}$$
$$= \tfrac{1}{2}\pi(a^2 + b^2 + c^2 - 2ch \text{ ct } \omega),$$
$$= \tfrac{1}{2}\pi(a^2 + b^2 + c^2 - 4\triangle \text{ ct } \omega).$$

Since ch is double the area of A B C. [*W. H. Echols.*]

3.

Two parallel straight lines are distant apart d; it is required to unite them by *two* circular arcs *of given radii* which shall have between them a common tangent of length t.

[*Elmo G. Harris.*]

SOLUTION.

Let L be (the length of the cross-over) the distance between the points of contact with the parallel tangents measured parallel to them.

Let R and r be the radii. Join the centers of the circles and call α the compliment of the angle which this line makes with t.

Then
$$\text{ta } \alpha = \frac{t}{R + r}.$$

It is easy to see that the central angles of the two arcs are equal, each represented by δ, say.

Then
$$L = (R + r) \text{ si } \delta + t \text{ co } \delta, \tag{1}$$
$$d = (R + r)(1 - \text{co } \delta) + t \text{ si } \delta, \tag{2}$$

or
$$(d - R - r) = t \text{ si } \delta - (R + r) \text{ co } \delta. \tag{3}$$

Square (1) and (3), add them and reduce the result to
$$L^2 - t^2 = 2d(R + r) - d^2.$$

This gives the relation between L and t, either may therefore be furnished with the data.

Also

$$\text{ta}\,(\alpha+\delta)=\frac{L}{R+r-d},$$

$$=\frac{\text{ta}\,\alpha+\text{ta}\,\delta}{1-\text{ta}\,\delta\,\text{ta}\,\alpha}.$$

Hence

$$\text{ta}\,\delta=\frac{L\,(R+r)-t}{R+r+t\,L}.$$

This solves the problem. If the radii are equal we have the familiar railway engineers' cross-over, and the results are

$$L^2-t^2=4d\,R-d^2,$$

$$\text{ta}\,\delta=\frac{2R\,L-t}{2R-t\,L}.$$

[*Elmo G. Harris.*]

[*Also by W. O. Whitescarner and Charles Puryear.*]

5.

Two straight lines O P and O Q are of lengths b' and a' respectively. From P a perpendicular P M is drawn to O Q and equal to it, cutting it at N. Show that the equation to the locus of P, as the point N moves on O Q and the point M on Q M, referred to O Q and O P as axes of x and y respectively, is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

[*W. H. Echols.*]

SOLUTION.

Let P′ N′ M′ be any position of the moving line. Let the angle between M O produced and O Q be ω. Find the equation to the locus of P′ referred to these as axes of y' and x' respectively. Thus, drawing the ordinate $P'A=y'$, the triangles O N′ M′ and N′ P′ A give

$$\frac{O\,N'+N'\,A}{N'\,A}=\frac{M'\,N'+N'\,P}{N'\,P'},$$

or

$$N'\,A=\frac{b'\,x'\,\text{si}\,\alpha}{a'},$$

where $\alpha = \text{P O Q}$.

The triangle N′ P′ A gives

$$\text{N}'\text{P}'^2 = b'^2 \text{ si}^2\alpha = y'^2 + \text{N}'\text{A}^2 + 2\text{N}'\text{A}\, y \text{ co } \omega,$$

$$= y'^2 + \frac{b'^2\, x'^2 \text{ si}^2\alpha}{a'^2} + \frac{2b' \text{ co } \omega \text{ si } \alpha\ \ x'\, y'}{a'},$$

or

$$y'^2 + \frac{b'^2 \text{ si }^2\alpha}{a'^2} x'^2 + \frac{2b' \text{ co } \omega \text{ si } \alpha}{a'} x'y' = b'^2 \text{ si}^2\alpha,$$

as the equation to the locus referred to O Q and O M. Transform this to the axes O P and O Q as y and x axes, by the transformation formulae

$$x' = x + y\frac{\text{si }(\omega - \alpha)}{\text{si } \omega}, \qquad y = y\frac{\text{si } \alpha}{\text{si } \omega},$$

and divide through by si $^2\alpha$.

Thus the equation to the locus is

$$a'^2\, b'^2 = \frac{a'^2}{\text{si}^2\omega} y^2 + b'^2 \left[x^2 + \frac{\text{si}^2(\omega - \alpha)}{\text{si}^2\omega} y^2 + 2\frac{\text{si }(\omega - \alpha)}{\text{si } \omega} x y\right]$$

$$+ 2a'b' \text{ co } \omega \left(\frac{x\, y}{\text{si } \omega} + y^2\, \frac{\text{si }(\omega - \alpha)}{\text{si}^2\omega}\right).$$

$$2 \text{ area O P M} = a'\, b' \text{ co } \alpha,$$

also

$$= \text{O M}\cdot\text{OP si }(\pi - \omega + \alpha),$$

or

$$= -\frac{b'^2 \text{ co } \alpha \text{ si }(\pi - \omega + \alpha)}{\text{co } \omega}.$$

Therefore

$$\text{si }(\omega - \alpha) = -\frac{a'}{b'} \text{ co } \omega.$$

Substituting this value in the equation above, it reduces readily to the required form

$$a'^2\, y^2 + b'^2\, x^2 = a'^2\, b'^2.$$

[*Charles P. Echols.*]

6.

Regarding the portion of the tangent to the hyperbola intercepted by the asymptotes as one diagonal of a square, what are the loci of the extremities of its other diagonal?

[*W. H. Echols.*]

SOLUTION.

Consider the point on the convex side of the curve, its distance from the tangent is evidently equal to the semi-diameter conjugate to that drawn through the point of contact (x', y') of the tangent to the hyperbola.

Let γ be the angle between these conjugate diameters of the hyperbola, whose center is O, and ρ the distance of the point whose locus is sought from O. Refer the locus to the axes or the hyperbola as coördinate axes.

The equation to the hyperbola is

$$b^2 x'^2 - a^2 y'^2 = a^2 b^2. \qquad (1)$$

The relations between the diameters are

$$\left.\begin{aligned} a'^2 + b'^2 &= a^2 + b^2, \\ a' b' \text{ si } \gamma &= a b. \end{aligned}\right\} \qquad (2)$$

From the triangle (o o, $x\,y$, $x'\,y'$) we have

$$\rho^2 = x^2 + y^2 = a'^2 + b'^2 - 2a' b' \text{ si } \gamma,$$

$$\therefore\ x^2 + y^2 = a'^2 + b'^2 - 2a\,b. \quad - - - - - - \qquad (3)$$

Combining (1), (2) and (3) we have

$$y'^2 = \frac{b^2}{2(a^2+b^2)}\,[x^2+y^2-(a-b)^2],$$

$$x'^2 = \frac{a^2}{2(a^2+b^2)}\,[x^2+y^2+(a+b)^2].$$

But $$(x-x')^2 + (y-y')^2 = b'^2,$$

and $$x'^2 + y'^2 = a'^2.$$

Hence $$2x\,x' + 2y\,y' = x^2 + y^2 + a^2 - b^2.$$

Which squared gives

$$x^2 x'^2+y^2-y'^2-\tfrac{1}{4}(x^2+y^2+a^2-b^2)^2=-2x\,y\,x'\,y'. \qquad (4)$$

The equation of the normal is

$$a^2 x\,y'+b^2 y\,x'=(a^2+b^2)\,x'\,y',$$

which when squared is

$$a^4 x^2 y'^2+b^4 y^2 x'^2-(a^2+b^2)^2 x'^2 y'^2=-2a^2 b^2 x\,y\,x'\,y'. \qquad (5)$$

Combining (4) and (5) to eliminate $x\,y\,x'\,y'$, and substituting in the resulting equation the values for x'^2 and y'^2 as obtained above we readily reduce the equation of the locus

$$\frac{x^2}{(a-b)^2}-\frac{y^2}{(a+b)^2}=1.$$

In like manner the equation to the locus of the other extremity of the diagonal would have been found to be

$$\frac{x^2}{(a+b)^2}-\frac{y^2}{(a-b)^2}=1.$$

[*W. H. Echols.*]

EXERCISES.

7.

On the sides of a triangle T, equilateral triangles are described, all outwards or all inwards. We thus get two new triangles T_1, T_2. Show that

(1). $$\Delta_1+\Delta_2=5\Delta,$$

where Δ, Δ_1, Δ_2 are the areas.

(2). The maximum inscribed ellipses of T_1 and T_2 are confocal.

[*Frank Morley.*]

8.

In the Cassinian $r r_1=k^2$ the angle between the central radius and one focal radius is equal to that between the other focal radius and the normal.

[*Frank Morley.*]

9.

Solve the equations

$$x^2+y z=a x+b c,$$
$$y^2+z x=b y+c a,$$
$$z^2+x y=c z+a b.$$

[*Frank Morley.*]

10.

A 100 foot steel tape is longer than standard, so that at a certain temperature the tape measures a horizontal chord of 100 standard feet under a pull of 16 pounds supported at its ends. Find the pull that will give 40, 50 and in general D (<100) standard foot horizontal chords, at same temperature, when the tape is supported at each end of the 40, 50, D foot graduations.

[*W. O. Whitescarver.*]

11.

A particle is set free at the highest point of a smooth sphere which stands on a horizontal plane. The particle slightly disturbed begins to move in a certain direction, where does it meet the plane and what is the duration of motion?

[*Elmo G. Harris.*]

12.

A smooth tube bent to the shape of a semi-ellipse is fixed in a vertical plane, its major axis horizontal, its semi-minor axis upward. A heavy flexible string passing through the tube and hanging ot rest is cut at one end of the tube. What is the velocity of the string as it leaves the tube? [*W. H. Echols.*]

13.

Given on the ground a circular curve of known radius intersecting a given straight line at a given point and given angle; it is required to unite the two by another circular curve of given radius. [*W. H. Echols.*]

14.

Given on the ground a circular curve of known radius intersecting a given straight line at a given point and given angle; it is required to unite the two by another circular curve of given radius in such a manner as to have a common tangent of length t between the curves. [*W. H. Echols.*]

15.

$$\int (a^2-x^2) \text{ arc-cos} \left(\frac{a}{2\sqrt{a^2-x^2}} \right) dx.$$

[*W. H. Echols.*]

SELECTED. CONSTRUCTIVE PROBLEMS IN ELEMENTARY GEOMETRY.

16.

Construct the triangle A B C which is similar to the given triangle L M N and which projects orthogonally upon a plane into the given triangle A′ B′ C′.

17.

Of the three concurrent edges *a*, *b*, *c* of a cube, the orthogonal projections on a plane *a*′, *b*′ of two are known, it is required to construct the projection of the cube.

18.

Of the three concurrent edges *a*, *b*, *c* of a cube, the orthogonal projection on a plane, *a*′ of one and the directions of the projections of the other two are known, it is required to construct the projection of the cube.

19.

Of the three concurrent edges *a*, *b*, *c* of a cube, the orthogonal projection on a plane *a*′ of one, the lengths of the orthogonal projections of the other two are known, it is required to construct the projection of the cube.

20.

Of the three concurrent edges *a*, *b*, *c* of a cube the orthogonal projection on a plane *a*′ of one, the length of the orthogonal projection of another and the direction of the projection of the third is known, it is required to construct the projection of the cube.

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