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Geometrical Researches

ON

THE THEORY OF PARALLELS,

BY

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BERLIN, 1840.

TRANSLATED FROM THE ORIGINAL

BY

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AUSTIN, 1891.

TRANSLATOR'S PREFACE.

Lobatschewsky was the first man ever to publish a non-Euclidian geometry.

Of the immortal essay now first appearing in English Gauss said, "the author has treated the matter with a master-hand and in the true geometer's spirit. I think I ought to call your attention to this book, whose perusal cannot fail to give you the most vivid pleasure."

Clifford says, "It is quite simple, merely Euclid without the vicious assumption, but the way things come out of one another is quite lovely." . . . "What Vesalius was to Galen, what Copernicus was to Ptolemy, that was Lobatschewsky to Euclid."

Says Sylvester: "In Quaternions the example has been given of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobatschewsky's of Geometry from Euclid's noted empirical axiom."

Cayley says, "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory, wherein this axiom was assumed not to hold good, or say a system of non-Euclidian plane geometry. There is a like system of non-Euclidian solid geometry."

GEORGE BRUCE HALSTEAD.

THEORY OF PARALLELS.

In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

As belonging to these imperfections, I consider the obscurity in the fundamental concepts of the geometrical magnitudes and in the manner and method of representing the measuring of these magnitudes, and finally the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have been so far in vain.

For this theory Legendre's endeavors have done nothing, since he was forced to leave the only rigid way, to turn into a side path, and take refuge in auxiliary theorems which he illogically strove to exhibit as necessary axioms. My first essay on the foundations of geometry I published in the Kasan *Messenger* for the year 1829. In the hope of having satisfied all requirements, I undertook hereupon a treatment of the whole of this science, and published my work in separate parts in the "*Gelehrten Schriften der Universitat Kasan*" for the years 1836, 1837, 1838, under the title "New Elements of Geometry, with a complete Theory of Parallels." The extent of this work perhaps hindered my countrymen from following such a subject, which since Legendre had lost its interest. Yet am I of the opinion, that the Theory of Parallels should not lose its claim to the attention of geometers, and therefore I aim to give here the substance of my investigations, remarking beforehand that contrary to the opinion of Legendre, all other imperfections, for example the definition of the straight line, show themselves foreign here and without any real influence on the theory of parallels.

In order not to fatigue my reader with the multitude of those theorems whose proofs present no difficulties, I prefix here only those of which a knowledge is necessary for what follows.

1. A straight line fits upon itself in all its positions. By this I mean, that during the revolution of the surface containing it the straight line does not change its place if it goes through two unmoving points in the surface: (*i. e.* if we turn the surface containing it about two points of the line, the line does not move.)

2. Two straight lines cannot intersect in two points.

3. A straight line sufficiently produced both ways must go out beyond all bounds, and in such way cuts a bounded plain into two parts.

4. Two straight lines perpendicular to a third, never intersect, how far soever they be produced.

5. A straight line always cuts another in going from one side of it over to the other side: (*i. e.* one straight line must cut another if it has points on both sides of it:)

6. Vertical angles, where the sides of one are productions of the sides of the other, are equal. This holds of plane rectilineal angles among themselves, as also of plane surface angles, (*i. e.* dihedral angles.)

7. Two straight lines cannot intersect, if a third cuts them at the same angle.

8. In a rectilineal triangle, equal sides lie opposite equal angles, and inversely.

9. In a rectilineal triangle, a greater side lies opposite a greater angle. In a right-angled triangle the hypothenuse is greater than either of the other sides, and the two angles adjacent to it are acute.

10. Rectilineal triangles are congruent if they have a side and two angles equal, or two sides and the included angle equal, or two sides and the angle opposite the greater equal, or three sides equal.

11. A straight line which stands at right angles upon two other straight lines not in one plane with it, is perpendicular to all straight lines drawn through the common intersection point in the plane of those two.

12. The intersection of a sphere with a plane is a circle.

13. A straight line at right angles to the intersection of two perpendicular planes, and in one, is perpendicular to the other.

14. In a spherical triangle, equal sides lie opposite equal angles, and inversely.

15. Spherical triangles are congruent, [or symmetrical], if they have two sides and the included angle equal, or a side and the adjacent angles equal.

From here follow the other theorems with their explanations and proofs.

16. All straight lines, which, in a plane, go out from a point, can with reference to a given straight line in the same plane, be divided into two classes, into *cutting* and *not-cutting*.

The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*.

From the point A (Fig. 1.) let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE.

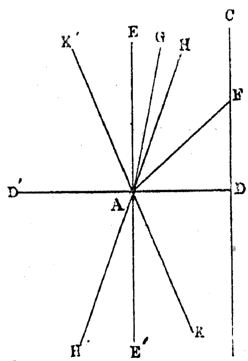

FIG. 1.

In the right angle EAD either will all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty, whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines, for example AG, which do not cut DC, how far so ever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel-angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD = p$.

If $\Pi(p)$ is a right angle, so will the prolongation AE′ of the perpendicular AE likewise be parallel to the prolongation DB of the line DC; in addition to which we remark, that in regard to the four right angles, which are made at the point A by the perpendiculars AE and AD, and their prolongations AE′ and AD′, every straight line which goes out from the point A, either itself, or at least its prolongation, lies in one of the two right angles which are turned toward BC, so that except the parallel, EE′, all others if they are sufficiently produced both ways, must intersect the line BC.

If $\Pi(p) < \frac{1}{2}\pi$, then upon the other side of AD, making the same angle $DAK = \Pi(p)$ will lie also a line AK, parallel to the prolongation DB of the line DC, so that under this assumption we must also make a distinction of *sides in parallelism.*

All remaining lines or their prolongations, within the two right angles turned toward BC, pertain to those that intersect, if they lie within the angle $HAK = 2\Pi(p)$ between the parallels; they pertain on the other hand to the non-intersecting, AG, if they lie upon the other sides of the parallels AH and AK, in the opening of the two angles $EAK = \frac{1}{2}\pi - \Pi(p)$, $E'AK = \frac{1}{2}\pi - \Pi(p)$, between the parallels and EE′ the perpendicular to AD. Upon the other side of the perpendicular EE′ will in like manner the prolongations AH′ and AK′ of the parallels AH and AK likewise be parallel to BC; the remaining lines pertain, if in the angle K'AH', to the intersecting, but if in the angles K'AE, H'AE' to the non-intersecting.

In accordance with this, for the assumption $\Pi(p) = \frac{1}{2}\pi$, the lines can be only intersecting or parallel; but if we assume that $\Pi(p) < \frac{1}{2}\pi$, then we must allow two parallels, one on the one and one on the other side; in addition we must distinguish the remaining lines into non-intersecting and intersecting.

For both assumptions it serves as the mark of parallelism that the line becomes intersecting for the smallest deviation toward the side where lies the parallel, so that if AH is parallel to DC, every line AF cuts DC, how small soever the angle HAF may be.

17. *A straight line maintains the characteristic of parallelism at all its points.*

Given AB (Fig 2.) parallel to CD, to which latter AC is per-

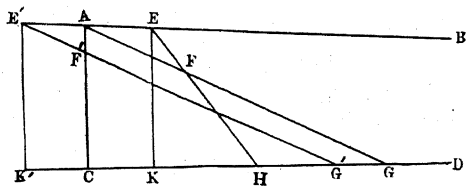

FIG. 2.

pendicular. We will consider two points taken at random on the line AB and its production beyond the perpendicular.

Let the point E lie on that side of the perpendicular on which AB is looked upon as parallel to CD.

Let fall from the point E a perpendicular EK on CD and so draw EF that it falls within the angle BEK.

Connect the points A and F by a straight line, whose production then (by Theorem 16) must cut CD somewhere in G. Thus we get a triangle ACG, into which the line EF goes; now since this latter, from the construction, cannot cut AC, and cannot cut AG or EK a second time (Theorem 2.) therefore it must meet CD somewhere at H (Theorem 3.)

Now let E′ be a point on the production of AB and E′K′ perpendicular to the production of the line CD; draw the line E′F′ making so small an angle AE′F′ that it cuts AC somewhere in F′; making the same angle with AB draw also from A the line AF whose production will cut CD in G, (Theorem 16.)

Thus we get a triangle AGC, into which goes the production of the line E′F′; since now this line cannot cut AE a second time, and also cannot cut AG, since the angle BAG = BE′O′, (Theorem 7) therefore must it meet CD somewhere in G′.

Therefore from whatever points E and E′ the lines EF and E′F′ go out, and however little they may diverge from the line AB, yet will they always cut CD, to which AB is parallel.

18. Two lines are always mutually parallel.

Let AC be a perpendicular on CD to which AB is parallel; if we draw from C the line CE making any acute angle ECD with CD, and let fall from A the perpendicular AF upon CE, we obtain a right-angled triangle ACF, in which AC, being the hypothenuse, is greater than the side AF, (Theorem 9.)

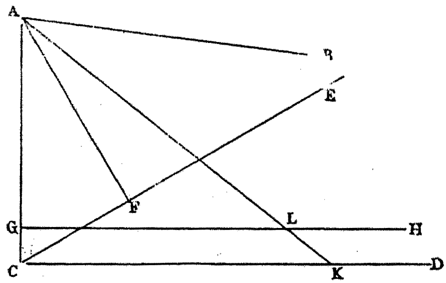

FIG. 3.

Make AG = AF, and slide the figure EFAB until AF coincides with AG, when AB and FE will take the position AK and GH, such that the angle BAK = FAC, consequently AK must cut the line DC somewhere in K, (Theorem (16), thus forming a triangle AKC, on one side of which the perpendicular GH intersects the line AK in L, (Theorem 3), and thus determines the distance AL of the intersection point of the lines AB and CE on the line AB from the point A.

Hence it follows, that CE will always intersect AB, how small soever may be the angle ECD, consequently CD is parallel to AB, (Theorem 16).

19. *In a rectilineal triangle the sum of the three angles cannot be greater than two right angles.*

Suppose in the triangle ABC (Figure 4.) the sum of the three angles is equal to $\pi+\alpha$; then choose in case of the inequality of the sides, the smallest BC, halve it in

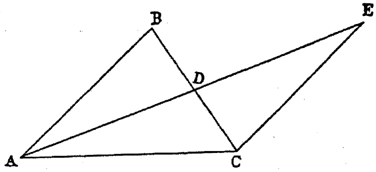

FIG. 4.

D, draw from A through D the line AD and make the prolongation of it, DE, equal to AD, then join the point E to the point C by the straight line EC. In the congruent triangles ADB and CDE, the angle ABD = DCE, and BAD = DEC, (Therems 6 and 10); whence follows that also in the triangle ACE the sum of the three angles must be equal to $\pi + \alpha$; but also the smallest angle BAC (Theorem 9), of the triangle ABC in passing over into the new triangle ACE has been cut up into the two parts EAC and AEC. Continuing this process, continually halving the side opposite the smallest angle, we must finally attain to a triangle in which the sum of the three angles is $\pi + \alpha$, but wherein are two angles, each of which, in absolute magnitude, is less than $\frac{1}{2}\alpha$; since now, however, the third angle cannot be greater than π, so must α be either null or negative.

20. *If in any rectilineal triangle the sum of the three angles is equal to two right angles, so is also the case for every other triangle.*

If in the rectilineal triangle ABC (Fig. 5.) the sum of the three angles $= \pi$, then must at least two of its angles, A and C, be acute. Let fall

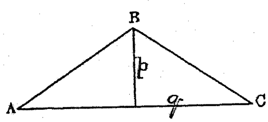

FIG. 5.

from the vertex of the third angle B upon the opposite side

AC the perpendicular p. This will cut the triangle into two right-angled triangles, in each of which the sum of three angles must also be π, since it cannot in either be greater than π, and in their combination not less than π.

So we obtain a right-angled triangle with the perpendicular sides p and q, and from this a quadrilateral whose opposite sides are equal and whose adjacent sides p and q are at right angles (Fig. 6).

By repetition of this quadrilateral we can make another with sides np and q, and finally a quadrilateral ABCD with sides at right angles to each other, such that AB = np,

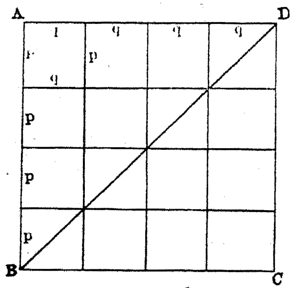

FIG. 6.

AD = mq, DC = np, BC = mq, where m and n are any whole numbers. Such a quadrilateral is divided by the diagonal DB into two congruent right angled triangles BAD and BCD, in each of which the sum of the three angles is $= \pi$.

The numbers n and m can be taken sufficiently great for the right-angled triangle ABC (Fig. 7.) whose perpendicular sides AB = np, BC = mq, to enclose within itself another given triangle BDE as soon as the right angles fit each other.

Drawing the line DC, we obtain right angled triangles of which every successive two have a side in common.

The triangle ABC is formed by the union of the two triangles ACD and DCB, in neither of which can the sum of the angles be greater than π; consequently it must be equal to π, in order that the sum in the compound triangle may be equal to π.

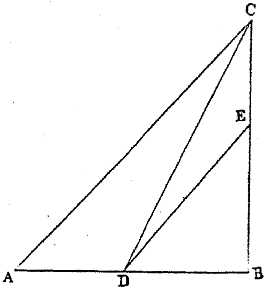

FIG. 7.

In the same way the triangle BDC consists of the two triangles DEC and DBE, consequently must in DBE the sum of the three angles be equal to π, and in general this must be true for every triangle since each can be cut into two right-angled triangles.

From this it follows that only two hypotheses are allowable: either is the sum of the three angles in all rectilineal triangles equal to π, or this sum is in all less than π.

21. *From a given point we can always draw a straight line that shall make with a given straight line an angle as small as we choose.*

Let fall from the given point A (Fig. 8.) upon the given line

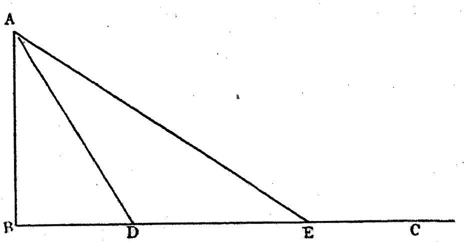

FIG. 8.

BC the perpendicular AB; take upon BC, at random, the point D; draw the line AD; make DE = AD, and draw AE.

In the right-angled triangle ABD let the angle ADB = α; then must in the isosceles triangle ADE the angle AED be either $\frac{1}{2}\alpha$ or less, (Theorems 8 and 20). Continuing thus we finally attain to such an angle AEB, as is less than any given angle.

22. *If two perpendiculars to the same straight line are parallel to each other, then the sum of the three angles in a rectilineal triangle is equal to two right angles.*

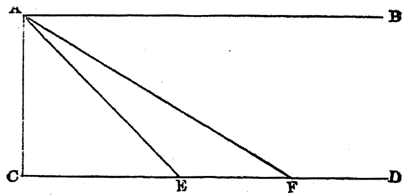

FIG. 9.

Let the lines AB and CD (Fig. 9.) be parallel to each other and perpendicular to AC.

Draw from A the lines AE and AF to the points E and F, which are taken on the line CD at any distances FC > EC from the point C.

Suppose in the right angled triangle ACE the sum of the three angles is equal to $\pi - \alpha$, in the triangle AEF equal to $\pi - \beta$, then must it in triangle ACF equal $\pi - \alpha - \beta$, where α and β cannot be negative.

Further, let the angle BAF = a, AFC = b, so is $\alpha + \beta = a - b$; now by revolving the line AF away from the perpendicular AC we can make the angle a between AF and the parallel AB as small as we choose; so also can we lessen the angle b, consequently the two angles α and β can have no other magnitude than $\alpha = 0$ and $\beta = 0$.

It follows that in all rectilineal triangles the sum of the three angles is either π and at the same time also the parallel angle $\Pi(p) = \frac{1}{2}\pi$ for every line p, or for all triangles this sum is $< \pi$ and at the same time also $\Pi(p) < \frac{1}{2}\pi$.

The first assumption serves as *foundation for the ordinary geometry and plane trigonometry.*

The second assumption can likewise be admitted without leading to any contradiction in the results, and founds a new geometric science, to which I have given the name, *Imaginary Geometry*, and which I intend here to expound as far as the development of the equations between the sides and angles of the rectilineal and spherical triangle.

23. *For every given angle* α *we can find a line* p, *such that* $\Pi(\mathrm{p}) = \alpha$.

Let AB and AC (Fig. 10.) be two straight lines which at the intersection-point A make the acute angle α; take at random on AB a point B′; from this point drop B′A′ at right angles to AC; make A′A″ = AA′; erect at A″ the perpendicular A″B″; and so continue until a perpendicular CD is attained,

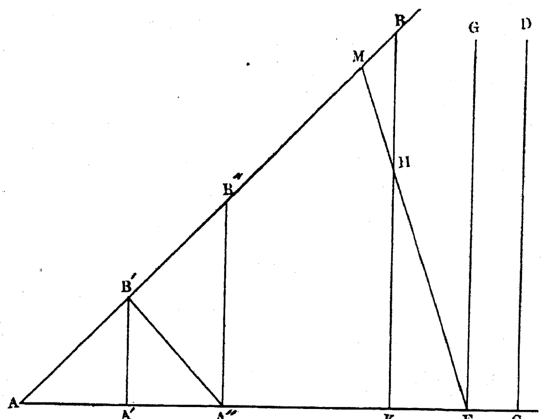

FIG. 10.

which no longer intersects AB. This must of necessity happen, for if in the triangle AA′B′ the sum of all three angles is equal to $\pi - a$, then in the triangle AB′A″ it equals $\pi - 2a$, in triangle AA″B″ less than $\pi - 2a$ (Theorem 20), and so forth, until it finally becomes negative and thereby shows the impossibility of constructing the triangle.

The perpendicular CD may be the very one nearer than which to the point A all others cut AB; at least in the passing over from those that cut to those not cutting such a perpendicular FG must exist.

Draw now from the point F the line FH, which makes with FG the acute angle HFG, on that side where lies the point A. From any point H of the line FH let fall upon AC the perpendicular HK, whose prolongation consequently must cut AB somewhere in B, and so makes a triangle AKB, into which the prolongation of the line FH enters, and therefore must meet the hypothenuse AB somewhere in M. Since the angle GFH is arbitrary, and can be taken as small as we wish, therefore FG is parallel to AB and AF = p (Theorems 16 and 18.)

One easily sees, that with the lessening of p the angle α increases, while, for p = 0, it approaches the value $\frac{1}{2}\pi$; with the growth of p the angle α decreases, while it continually approaches zero for p = ∞.

Since we are wholly at liberty to choose what angle we will understand by the symbol $\Pi(p)$ when the line p is expressed by a negative number, so we will assume

$$\Pi(p) + \Pi(-p) = \pi.$$

an equation which shall hold for all values of p, positive as well as negative, and for p = 0.

24. *The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another.*

If to the line AB (Fig. 11.) two perpendiculars AC = BE are erected, and their end-points C and E joined by a straight line, then will the quadrilateral CABE have two right angles at A and B, but two acute angles at C and E (Theorem 22,) which are equal to one another, as we can easily

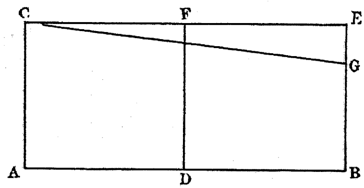

FIG 11.

see by thinking the quadrilateral superimposed upon itself so that the line BE falls upon AC and AC upon BE.

Halve AB and erect at the mid-point D the line DF perpendicular to AB. This line must also be perpendicular to CE, since the quadrilaterals CADF and FDBE fit one another if we so place one on the other that the line DF remains in the same position. Hence the line CE cannot be parallel to AB, but the parallel to AB for the point C, namely CG, must incline toward AB (Theorem 16), and cut from the perpendicular BE a part BG< CA.

Since C is a random point in the line CG, it follows that CG itself nears AB the more the farther it is prolonged.

25. *Two straight lines which are parallel to a third, are also parallel to one another.*

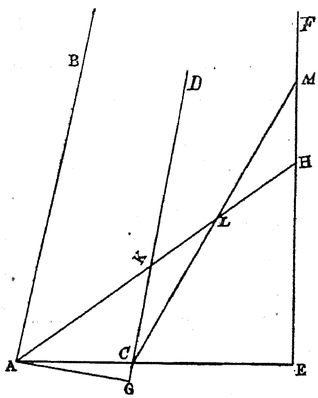

FIG. 12.

We will first assume that the three lines AB, CD, EF, (Fig. 12.) lie in one plane. If two of them in order AB and CD are parallel to the outmost one EF, so are AB and CD parallel to one another. In order to prove this, let fall from any point A of the outer line AB, upon the other outer line FE, the perpendicular AE, which will cut the middle line CD in some point C

(Theorem 3), at an angle DCE< $\frac{1}{2}\pi$ on the side toward EF the parallel to CD (Theorem 22).

A perpendicular AG let fall upon CD from the same point A, must fall within the opening of the acute angle ACG (Theorem 9); every other line AH from A drawn within the angle BAC, must cut EF, the parallel to AB, somewhere in H, how small soever the angle BAH may be; consequently will CD in the triangle AEH cut the line AH somewhere in K, since it is impossible that it should meet EF. If AH from the point A went out within the angle CAG, then must it cut the prolongation of CD between the points C and G in the triangle CAG. Hence follows, that AB and CD are parallel (Theorems 16 and 18).

Were both the outer lines AB and EF assumed parallel to the middle line CD, so would every line AK from the point A, drawn within the angle BAE, cut the line CD somewhere in the point K, how small soever the angle BAK might be.

Upon the prolongation of AK take at random a point L and join it with C by the line CL, which must cut EF somewhere in M, thus making a triangle MCE.

The prolongation of the line AL within the triangle MCE can cut neither AC nor CM a second time, consequently it must meet EF somewhere in H; therefore AB and EF are mutually parallel.

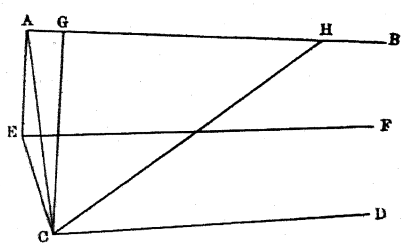

FIG. 13.

Let now the parallels AB and CD (Fig. 13.) lie in two planes whose intersection line is EF. From a random point E of this latter let fall a perpendicular EA upon one of the two parallels, *e. g.*, upon AB, then from A the foot of the perpendicular EA, let fall a new perpendicular AC upon the othe rparallel CD and join the end-points E and C of the two perpendiculars by the line EC. The angle BAC must be acute (Theorem 22), consequently a perpendicular CG from C let fall upon AB meets it in the point G upon that side of CA on which the lines AB and CD are considered as parallel.

Every line EH [in the plane FEAB]. however little it diverges from EF, pertains with the line EC to a plane which must cut the plane of the two parallels AB and CD along some line CH. This latter line cuts AB somewhere, and in fact in the very point H which is common to all three planes, through which necessarily also the line EH goes; consequently EF is parallel to AB.

In the same way we may show the parallelism of EF and CD.

Therefore the hypothesis, that a line EF is parallel to one of two other parallels, AB and CD, is the same as considering EF as the intersection of two planes in which two parallels AB, CD, lie.

Consequently two lines are parallel to one another if they are parallel to a third line, though the three be not co-planar

The last theorem can be thus expressed:

Three planes intersect in lines which are all parallel to each other if the parallelism of two is pre-supposed.

26. *Triangles standing opposite to one another on the sphere are equivalent in surface.*

By opposite triangles we here understand such as are made on both sides of the ceuter by the intersections of the sphere with planes; in such triangles therefore the sides and angles are in contrary order.

In the opposite triangles ABC and A'B'C' (Fig. 14., where one of them must be looked upon as represented turned about), we have the sides AB = A′B′, BC = B′C′, CA = C′A′ and the corresponding angles at the points A, B, C, are likewise equal

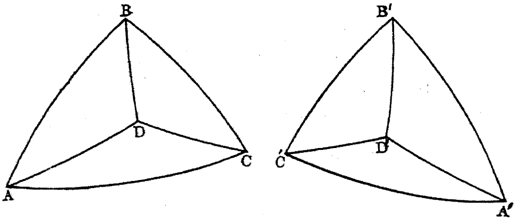

FIG. 14.

to those in the other triangle at the points A′, B′, C′.

Through the three points A, B, C, suppose a plane passed and upon it from the center of the sphere a perpendicular dropped, whose prolongations both ways cut both opposite triangles in the points D and D′ of the sphere. The distances of the first D from the points A B C, in arcs of great circles on the sphere, must be equal (Theorem 12), as well to each other as also to the distances D′A′, D′B′, D′C′, on the other triangle (Theorem 6), consequently the isosceles triangles about the points D and D′ in the two spherical triangles ABC and A′B′C′ are congruent.

In order to judge of the equivalence of any two surfaces in general I take the following theorem as fundamental:

Two surfaces are equivalent when they arise from the mating or separating of equal parts.

27. *A three-sided solid angle equals the half sum of the surface angles less a right angle.*

In the spherical triangle ABC (Fig. 15.), where each side $< \pi$, designate the angles by A, B, C; prolong the side AB so that a whole circle ABA′B′A is produced; this divides the sphere into two equal parts.

In that half in which is the triangle ABC, prolong now the other two sides through their common intersection point C, until they meet the circle in A′ and B′.

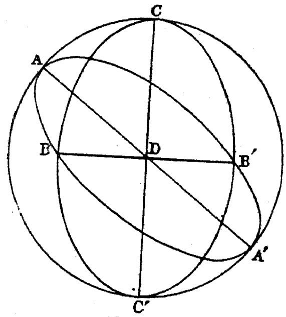

FIG. 15.

In this way the hemisphere is divided into four triangles, ABC, ACB′, B′CA′ A′CB, whose size may be designated by P, X, Y, Z. It is evident that here $P+X=B$, $P+Z=A$.

The size of the spherical triangle Y equals that of the opposite triangle ABC′, having a side AB in common with the triangle P, and whose third angle C′ lies at the end-point of the diameter of the sphere which goes from C through the center D of the sphere (Theorem 26). Hence it follows that $P+Y=C$, and since $P+X+Y+Z=\pi$, therefore we have also,

$$P=\tfrac{1}{2}(A+B+C-\pi).$$

We may attain to the same conclusion in another way, based solely upon the theorem about the equivalence of surfaces given above, (Theorem 26).

In the spherical triangle ABC (Fig. 16.) halve the sides AB and BC, and through the mid-points D and E draw a great circle; upon this let fall from A, B, C, the perpendiculars AF, BH, and CG. If the perpendicular from B falls at H between D and

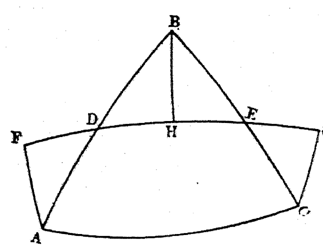

FIG. 16.

E, then will of the triangles so made BDH = AFD, and BHE = EGC, (Theorems 6 and 15), whence follows that the surface of the triangle ABC equals that of the quadrilateral AFGC (Theorem 16).

If the point H coincides with the middle point E of the side BC (Fig. 17.), only two equal right angled triangles AFD and BDE are made, by whose interchange the equivalence of the surfaces of the triangle ABC and the quadrilateral AFEC is established.

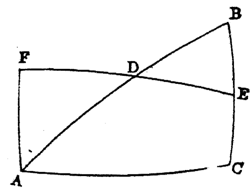

FIG. 17.

If, finally, the point H falls outside the triangle ABC, (Fig. 18), the perpendicular CG, goes, in consequence, through the triangle, and so we go over from the triangle ABC to the quad-

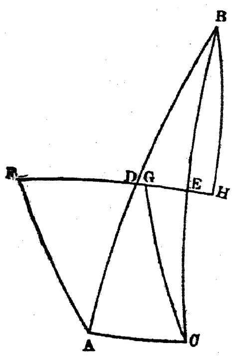

FIG. 18.

rilateral AFGC by adding the triangle FAD = DBH, and then taking away the triangle CGE = EBH.

Supposing in the spherical quadrilateral AEGC a great circle passed through the points A and G, as also through F and C, then will their arcs between AG and FC equal one another, (Theorem 15), consequently also the triangles FAC and ACG be congruent (Theorem 15), and the angle FAC equal the angle ACG.

Hence follows, that in all the preceding cases, the sum of all three angles of the spherical triangle equals the sum of the two equal angles in the quadrilateral which are not the right angles.

Therefore we can, for every spherical triangle, in which the sum of the three angles is S, find a quadrilateral with equivalent surface, in which are two right angles and two equal perpendicular sides, and where the two other angles are each $\frac{1}{2}$S.

Let now ABCD (Fig. 19.) be the spherical quadrilateral, where the sides AB = DC are perpendicular to BC, and the angles A and D each $\frac{1}{2}$S.

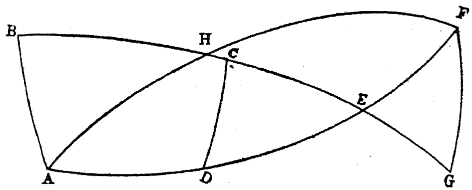

FIG. 19

Prolong the sides AD and BC until they cut one another in E, and further beyond E, make DE = EF and let fall upon the prolongation of BC the perpendicular FG. Bisect the whole arc BG and join the mid-point H by great-circle-arcs with A and F.

The triangles EFG and DCE are congruent (Theorem 15), so FG = DC = AB.

The triangles ABH and HGF are likewise congruent, since they are right angled and have equal perpendicular sides, consequently AH and AF pertain to *one* circle, the arc AHF = π, ADEF likewise = π, the angle HAD = HFE = $\frac{1}{2}$S − BAH = $\frac{1}{2}$S − HFG = $\frac{1}{2}$S − HFE − HFG = $\frac{1}{2}$S − HAD − π + $\frac{1}{2}$S; consequently, angle HFE = $\frac{1}{2}(S-\pi)$; or what is the same, this equals the size of the lune AHFDA, which again is equal to the quadrilateral ABCD, as we easily see if we pass over from the one to the other by first adding the triangle EFG and then BAH and thereupon taking away the triangles equal to them DCE and HFG.

Therefore $\frac{1}{2}(S-\pi)$ is the size of the quadrilateral ABCD and at the same time also that of the spherical triangle in which the sum of the three angles is equal to S.

28. *If three planes cut each other in parallel lines, then the sum of the three surface angles equals two right angles.*

Let AA′, BB′ CC′ (Fig. 20.) be three parallels made by the intersection of planes (Theorem 25). Take upon them at random three points A, B, C, and suppose through these a plane

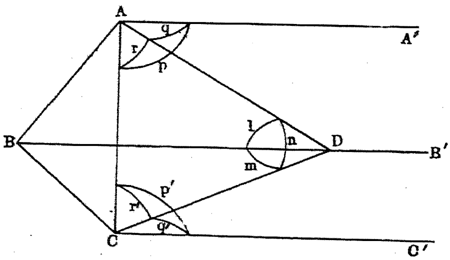

FIG. 20.

passed, which consequently will cut the planes of the parallels along the straight lines AB, AC, and BC. Further, pass through the line AC and any point D on the BB′, another plane, whose intersection with the two planes of the parallels AA′ and BB′, CC′ and BB′ produces the two lines AD and DC, and whose inclination to the third plane of the parallels AA′ and CC′ we will designate by w.

The angles between the three planes in which the parallels lie will be designated by X, Y, Z, respectively at the lines AA′, BB′, CC′; finally call the linear angles BDC $= a$, ADC $= b$, ADB $= c$.

About A as center suppose a sphere described, upon which the intersections of the straight lines AC, AD AA′ with it determine a spherical triangle, with the sides p, q, and r. Call its size α. Opposite the side q lies the angle w, opposite r lies X, and consequently opposite p lies the angle $\pi+2\alpha-w-X$, (Theorem 27).

In like manner CA, CD, CC′ cut a sphere about the center C, and determining a triangle of size β, with the angles p′, q′, r′, and the angles, w opposite q′, Z opposite r′, and consequently $\pi+2\beta-w-Z$ opposite p′.

Finally is determined by the intersection of a sphere about D with the lines DA, DB, DC, a spherical triangle, whose sides are l, m, n, and the angles opposite them $w+Z-2\beta$, $w+X-2\alpha$, and Y. Consequently its size $\delta=\frac{1}{2}(X+Y+Z-\pi)-\alpha-\beta+w$.

Decreasing w lessens also the size of the triangles α and β, so that $\alpha+\beta-w$ can be made smaller than any given number.

In the triangle δ can likewise the sides l and m be lessened even to vanishing (Theorem 21), consequently the triangle δ can be placed with one of its sides l or m upon a great circle of the sphere as often as you choose without thereby filling up the half of the sphere, hence δ vanishes together with w; whence follows that necessarily we must have

$$X+Y+Z=\pi.$$

29. *In a rectilineal triangle, the perpendiculars erected at the mid-points of the sides either do not meet, or they all three cut each other in one point.*

Having pre-supposed in the triangle ABC (Fig. 21), that the two perpendiculars ED and DF, which are erected upon the sides AB and BC at their mid-points E and F, intersect in the point D, then draw within the angles of the triangle the lines DA, DB, DC.

In the congruent triangles ADE and BDE (Theorem 10), we have AD = BD, thus follows also that BD = CD; the triangle ADC is hence isosceles, consequently the perpendicular dropped from the vertex D upon the base AC falls upon G the mid-point of the base.

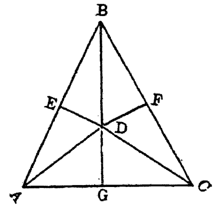

FIG. 21.

The proof remains unchanged also in the case when the intersection point D of the two perpendiculars ED and FD falls in the line AC itself. or falls without the triangle.

In case we therefore presuppose that two of those perpendiculars do not intersect, then also the third cannot meet with them.

30. *The perpendiculars which are erected upon the sides of a rectilineal triangle at their mid-points, must all three be parallel to each other, so soon as the parallelism of two of them is presupposed.*

In the triangle ABC (Fig. 22.) let the lines DE, FG, HK, be erected perpendicular upon the sides at their mid-points D, F, H. We will in the first place assume that the two perpendiculars DE and FG are parallel, cutting the line AB in L and M, and that the perpendicular HK lies between them. Within the angle BLE draw from the point L at random, a straight line LG, which must cut FG somewhere in G, how small soever the angle of deviation GLE may be. (Theorem 16).

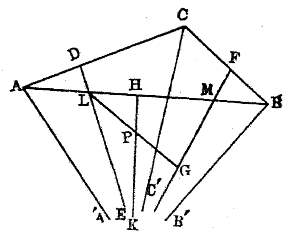

FIG. 22.

Since in the triangle LGM the perpendicular HK cannot meet with MG (Theorem 29), therefore it must cut LG somewhere in P, whence follows. that HK is parallel to DE (Theorem 16), and to MG (Theorems 18 and 25).

Put in the side BC = 2a, AC = 2b, AB = 2c, and designate the angles opposite these sides by A, B, C, then we have in the case just considered

$$A = \Pi(b) - \Pi(c),$$
$$B = \Pi(a) - \Pi(c),$$
$$C = \Pi(a) + \Pi(b),$$

as one may easily show with help of the lines AA', BB', CC', which are drawn from the points A, B, C, parallel to the perpendicular HK and consequently to both the other perpendiculars DE and FG, (Theorems 23 and 25).

Let now the two perpendiculars HK and FG be parallel, then can the third DE not cut them (Theorem 29), hence is it either parallel to them, or it cuts AA'.

The last assumption is not other than that the angle

$$c > \Pi(a) + \Pi(b.)$$

If we lessen this angle, so that it becomes equal to $\Pi(a) + \Pi(b)$, while we in that way give the line AC the new position CQ, (Fig. 23), and designate the size of the third side BQ by $2c'$, then must the angle CBQ at the point B, which is increased, in accordance with what is proved above, be equal to

$$\Pi(a) - \Pi(c') > \Pi(a) - \Pi(c),$$

whence follows $c' > c$ (Theorem 23).

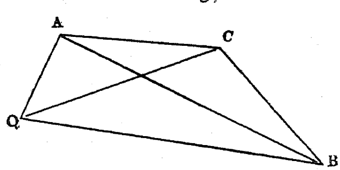

FIG. 23.

In the triangle ACQ are, however, the angles at A and Q equal, hence in the triangle ABQ must the angle at Q be greater than that at the point A, consequently is $AB > BQ$, (Theorem 9); that is $c > c'$.

31. *We call boundary line (oricycle) that curve line lying in a plane for which all perpendiculars erected at the mid-points of chords are parallel to each other.*

In conformity with this definition we can represent the generation of a boundary line, if we draw to a given line AB

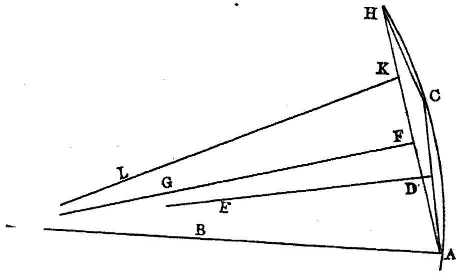

FIG. 24.

(Fig 24.) from a given point A in it, making different angles CAB $= \Pi(a)$, chords AC $= 2a$; the end C of such a chord will lie on the boundary line, whose points we can thus gradually determine.

The perpendicular DE erected upon the chord AC at its mid-point D will be parallel to the line AB, which we will call the *Axis of the boundary line.* In like manner will also each perpendicular FG erected at the mid-point of any chord AH, be parallel to AB, consequently must this peculiarity also pertain to every perpendicular KL in general which is erected at the mid-point K of any chord CH, between whatever points C and H of the boundary line this may be drawn (Theorem 30). Such perpendiculars must therefore likewise, without distinction from AB, be called *Axes of the boundary line.*

32. *A circle with continually increasing radius merges into the boundary line.*

Given AB (Fig. 25.) a chord of the boundary line; draw from the end-points A and B of the chord two axes AC and BF′, which consequently will make with the chord two equal angles BAC $=$ ABF′ $= \alpha$ (Theorem 31).

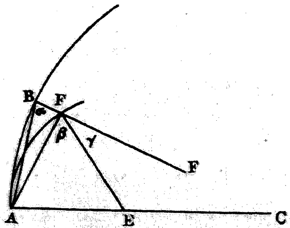

FIG. 25.

Upon one of these axes AC, take

anywhere the point E as center of a circle, and draw the arc AF from the initial point A of the axis AC to its intersection point F with the other axis BF′.

The radius of the circle, FE, corresponding to the point F will make on the one side with the chord AF an angle AFE $= \beta$, nnd on the other side with the axis BF′ the angle EFF′ $= \gamma$. It follows that the angle between the two chords BAF $= \alpha - \beta < \beta + \gamma - \alpha$ (Theorem 22); whence follows, $\alpha - \beta < \frac{1}{2}\gamma$.

Since now however the angle γ approaches the limit zero, as well in consequence of a moving of the center E in the direction AC, when F remains unchanged, (Theorem 21), as also in consequence of an approach of F to B on the axis BF, when the center E remains in its position (Theorem 22), so it follows, that with such a lessening of the angle γ, also the angle $\alpha - \beta$, or the mutual inclination of the two chords AB and AF, and hence also the distance of the point B on the boundary-line from the point F on the circle, tends to vanish.

Consequently one may also call the boundary-line *a circle with infinitely great radius.*

33. Let AA′ = BB′ = x (Figure 26), be two lines parallel toward the side from A to A′, which parallels serve as axes for the two boundary arcs (arcs on two boundary lines) AB = s, A′B′ = s', then is

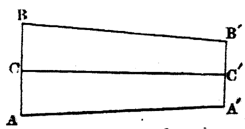

FIG. 26.

$$s' = s\mathbf{e}^{x} \quad *$$

where $\mathbf{e}$ is independent of the arcs s, s' and of the straight lines x, the distance of the arc s' from s.

In order to prove this, assume that the ratio of the arc s to s' is equal to the ratio of the two whole numbers l and m.

Between the two axes AA′, BB′ draw yet a third axis CC′, which so cuts off from the arc AB a part AC = t and from the

* For this equation read $s' = s\mathbf{e}^{-x}$

arc A'B' on the same side. a part A'C' $= t'$. Assume the ratio of t to s equal to that of the whole numbers p and q, so that

$$s = \frac{n}{m} s', \quad t = \frac{p}{q} s.$$

Divide now s by axes into nq equal parts, then will there be mq such parts on s' and np on t.

However there correspond to these equal parts on s and t likewise equal parts on s' and t', consequently we have

$$\frac{t'}{t} = \frac{s'}{s}.$$

Hence also wherever the two arcs t and t' may be taken between the two axes AA' and BB', the ratio of t to t' remains always the same, as long as the distance x between them remains the same. If we therefore for $x = 1$, put $s = \mathbf{e}s'$, then we must have for every x

$$s' = s\mathbf{e}^{-x}.$$

Since $\mathbf{e}$ is an unknown number only subjected to the condition $\mathbf{e} > 1$, and further the linear unit for x may be taken at will, therefore we may, for the simplification of reckoning, so choose it that by $\mathbf{e}$ is to be understood the base of the Napierian logarithms.

We may here remark, that $s' = 0$ for $x = \infty$, hence not only does the distance between two parallels decrease (Theorem 24), but with the prolongation of the parallels toward the side of the parallelism this at last wholly vanishes, Parallel lines have therefore the character of asymptotes.

34. *Boundary surface* (*orisphere*) we call that surface which arises from the revolution of the boundary line about one of its axes, which, together with all other axes of the boundary-line, will be also an axis of the boundary-surface.

A chord is inclined at equal angles to such axes drawn through its end-points, wheresoever these two end-points may be taken on the boundary-surface.

Let A, B, C, (Fig. 27.), be three points on the boundary-surface; AA', the axis of revolution, BB′ and CC′ two other

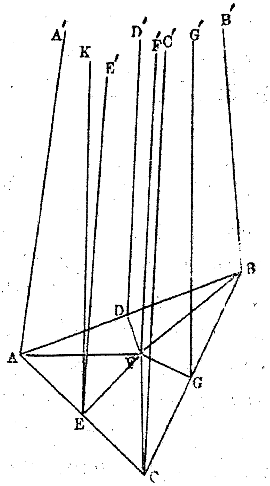

FIG 27.

axes, hence AB and AC chords to which the axes are iuclined at equal angles A′AB = B′BA, A′AC = C′CA (Theorem 31.)

Two axes BB′, CC′, drawn through the end-points of the third chord BC, are likewise parallel and lie in one plane, (Theorem 25).

A perpendicular DD′ erected at the mid-point D of the chord AB and in the plane of the two parallels AA′, BB′, must be parallel to the three axes AA′, BB′, CC′, (Theorems 23 and 25); just such a perpendicular EE′ upon the chord AC in the plane of the parallels AA′, CC′ will be parallel to the three axes AA′, BB′, CC′, and the perpendicular DD′. Let now the angle between the plane in which the parallels AA′ and BB′ lie, and the plane of the triangle ABC be designated by $\Pi(a)$, where a may be positive, negative, or null. If a is positive, then erect FD $= a$ within the triangle ABC, and in its plane, perpendicular upon the chord AB at its mid-point D.

Were a a negative number, then must FD $= a$ be drawn outside the triangle on the other side of the chord AB; when $a = 0$, the point F coincides with D.

In all cases arise two congruent right-angled triangles AFD and DFB, consequently we have FA = FB.

Erect now at F the line FF′ perpendicular to the plane of the triangle ABC.

Since the angle D′DF $= \Pi(a)$, and DF $= a$, so FF′ is parallel to DD′ and the line EE′, with which also it lies in one plane perpendicular to the plane of the triangle ABC.

Suppose now in the plane of the parallels EE′, FF′ upon EF the perpendicular EK erected, then will this be also at right angles to the plane of the triangle ABC, (Theorem 13), and to the line AE lying in this plane, Theorem 11); and consequently must AE, which is perpendicular to EK and EE′, be also at the same time perpendicular to FE, (Theorem 11). The triangles AEF and FEC are congruent, since they are right-angled and have the sides about the right angles equal, hence is

AF = FC = FB.

A perpendicular from the vertex F of the isosceles triangle BFC let fall upon the base BC, goes through its mid-point G; a plane passed through this perpendicular FG and the line FF′ must be perpendicular to the plane of the triangle ABC, and cuts the plane of the parallels BB′, CC′ along the line GG′, which is likewise parallel to BB′ and CC′, (Theorem 25); since now CG is at right angles to FG, and hence at the same time also to GG′, so consequently is the angle C′CG = B′BG, (Theorem 23)

Hence follows, that for the boundary-surface each of the axes may be considered as axis of revolution.

Principal-plane we will call each plane passed through an axis of the boundary surface.

Accordingly every *Principal-plane* cuts the boundary-surface in the boundary line, while for another position of the cutting plane this intersection is a circle.

Three principal planes which mutually cut each other, make with each other angles whose sum is π, (Theorem 28).

These angles we will consider as angles in the boundary-triangle whose sides are arcs of the boundary-line, which are made on the boundary surface by the intersections with the three principal-planes. Consequently the same interdependence of the angles and sides pertains to the boundary-triangles, that is proved in the ordinary geometry for the rectilineal triangle.

35. In what follows, we will designate the size of a line by a letter with an accent added, *e. g.* x', in order to indicate that this has a relation to that of another line, which is represented by the same letter without accent x, which relation is given by the equation

$$\Pi(x)+\Pi(x')=\tfrac{1}{2}\pi.$$

Let now ABC (Fig. 28.) be a rectilineal right angled triangle, where the hypothenuse AB $= c$, the other sides AC $= b$,

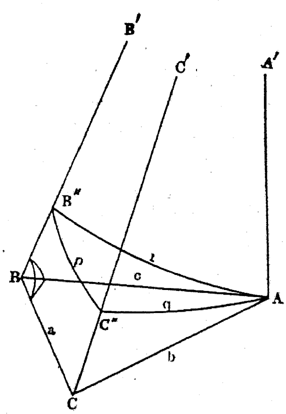

FIG. 28.

BC $= a$, and the angles opposite them are

$$\text{BAC} = \Pi(a), \quad \text{ABC} = \Pi(\beta).$$

At the point A erect the line AA′ at right angles to the plane of the triangle ABC, and from the points B and C draw BB′ and CC′ parallel to AA′.

The planes in which these three parallels lie make with each other the angles: $\Pi(\alpha)$ at AA′, a right angle at CC′ (Theorems 11 and 13), consequently $\Pi(\alpha')$ at BB′ (Theorem 28.)

The intersections of the lines BA, BC, BB′ with a sphere described about the point B as center, determine a spherical triangle mnk, in which the sides are $mn = \Pi(c)$, $kn = \Pi(\beta)$, $mk = \Pi(a)$ and the opposite angles are $\Pi(b)$, $\Pi(\alpha')$, $\frac{1}{2}\pi$.

Therefore we must, with the existence of a rectilineal triangle whose sides are a, b, c, and the opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{1}{2}\pi$, also admit the existence of a spherical triangle (Fig. 29.) with the side $\Pi(c)$, $\Pi(\beta)$, $\Pi(a)$ and the opposite angles $\Pi(b)$, $\Pi(\alpha')$, $\frac{1}{2}\pi$.

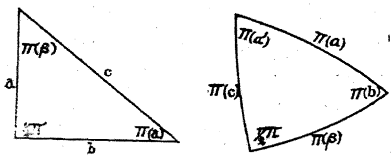

FIG. 29.

Of these two triangles, however, also inversely the existence of the spherical triangle necessitates anew that of a rectilineal, which in consequence, also can have the sides a, α', β, and the opposite angles $\Pi(b')$ $\Pi(c)$, $\frac{1}{2}\pi$.

Hence we may pass over from a, b, c, α, β, to b, a, c, β, α, and also to a, α', β, b', c.

Suppose through the point A (Fig. 28.) with AA′ as axis, a boundary-surface passed, which cuts the two other axes BB′, CC′, in B″ and C″, and whose intersections with the planes of the parallels form a boundary-triangle, whose sides are B″C″ $= p$, C″A $= q$, B″A $= r$, and the angles opposite them $\Pi(\alpha)$, $\Pi(\alpha')$, $\frac{1}{2}\pi$, and where consequently (Theorem 34):

$$p = r \sin \Pi(\alpha),\ q = r \cos \Pi(\alpha).$$

Now break the connection of the three principal-planes along the line BB′, and turn them out from each other so that they with all the lines lying in them come to lie in one plane, where consequently the arcs p, q, r will unite to a single arc of a boundary-line, which goes through the point A and has AA′ for axis, in such a manner that (Fig. 30.) on the one side will

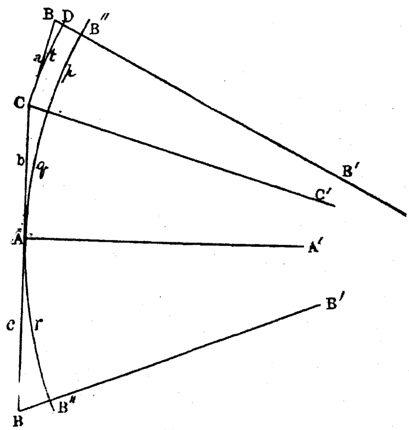

FIG. 30.

lie, the arcs q and p, the side b of the triangle, which is perpendicular to AA′ at A, the axis CC′ going from the end of b parallel to AA′ and through C″ the union-point of p and q, the side a perpendicular to CC′ at the point C, and from the end-point of a the axis BB′ parallel to AA′ which goes through the end-point B″ of the arc p.

On the other side of AA′ will lie, the side c perpendicular to AA′ at the point A, and the axis BB′ parallel to AA′, and going through the end-point B″ of the arc r remote from the end-point of b.

The size of the line CC″ depends upon b, which dependence we will express by CC″ $= f(b)$.

In like manner we will have BB″ $= f(b)$.

If we describe, taking CC′ as axis, a new boundary line from the point C to its intersection D with the axis BB′ and designate the arc CD by t, then is $BD = f(a)$.

$$BB'' = BD + DB'' = BD + CC'', \text{ consequently}$$

$$f(c) = f(a) + f(b).$$

Moreover, we perceive, that (Theorem 32)

$$t = p\mathbf{e}^{f(b)} = r \sin \Pi(\alpha)\ \mathbf{e}^{f(b)}.$$

If the perpendicular to the plane of the triangle ABC (Fig. 28.) were erected at B instead of at the point A, then would the lines c and r remain the same, the arcs q and t would change to t and q, the straight lines a and b into b and a, and the angle $\Pi(\alpha)$ into $\Pi(\beta)$, consequently we would have

$$q = r \sin \Pi(\beta)\ \mathbf{e}^{f(a)},$$

whence follows by substituting the value of q,

$$\cos \Pi(\alpha) = \sin \Pi(\beta)\ \mathbf{e}^{f(a)},$$

and if we change α and β into b' and c,

$$\sin \Pi(b) = \sin \Pi(c)\ \mathbf{e}^{f(a)};$$

further, by multiplication with $\mathbf{e}^{f(b)}$

$$\sin \Pi(b)\ \mathbf{e}^{f(b)} = \sin \Pi(c)\ \mathbf{e}^{f(c)}.$$

Hence follows also

$$\sin \Pi(a)\ \mathbf{e}^{f(a)} = \sin \Pi(b)\ \mathbf{e}^{f(b)}.$$

Since now, however, the straight lines a and b are independent of one another, and moreover, for $b=0$, $f(b)=0$, $\Pi(b)=\frac{1}{2}\pi$, so we have for every straight line a

$$\mathbf{e}^{-f(a)} = \sin \Pi(a).$$

Therefore,

$$\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b),$$
$$\sin \Pi(b) = \cos \Pi(\alpha) \sin \Pi(a).$$

Hence we obtain besides by mutation of the letters

$$\sin \Pi(\alpha) = \cos \Pi(\beta) \sin \Pi(b),$$
$$\cos \Pi(b) = \cos \Pi(c) \cos \Pi(\alpha),$$
$$\cos \Pi(a) = \cos \Pi(c) \cos \Pi(\beta).$$

If we designate in the right-angled spherical triangle (Fig. 29) the sides $\Pi(c)$, $\Pi(\beta)$, $\Pi(a)$, with the opposite angles $\Pi(b)$, $\Pi(\alpha')$, by the letters a, b, c, A, B, then the obtained equations take on the form of those which we know as proved in spherical trigonometry for the right-angled triangle, namely,

$$\sin a = \sin c \sin \mathrm{A},$$

$$\sin b = \sin c \sin \mathrm{B},$$

$$\cos \mathrm{A} = \cos a \sin \mathrm{B},$$

$$\cos \mathrm{B} = \cos b, \sin \mathrm{A},$$

$$\cos c = \cos a, \cos b;$$

from which equations we can pass over to those for all spherical triangles in general.

Hence spherical trigonometry is not dependent upon whether in a rectilineal triangle the sum of the three angles is equal to two right angles or not.

36. We will now consider anew the right-angled rectilineal triangle ABC (Fig. 31), in which the sides are a, b, c, and the opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\frac{1}{2}\pi$.

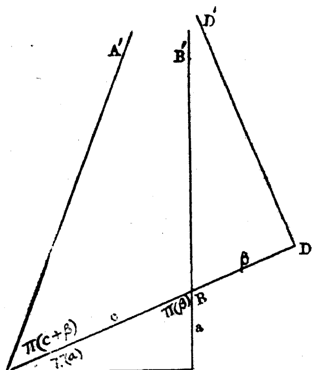

FIG. 31.

Prolong the hypothenuse c through the point B, and make BD$=\beta$; at the point D erect upon BD the perpendicular DD', which consequently will be parallel to BB', the prolongation of the side a beyond the point B. Parallel to DD' from the point A draw AA', which is at the same time also parallel to CB', (Theorem 25), therefore is the angle

$$A'AD = \Pi(c+\beta),$$
$$A'AC = \Pi(b), \quad \text{consequently}$$
$$\Pi(b) = \Pi(a) + \Pi(c+\beta).$$

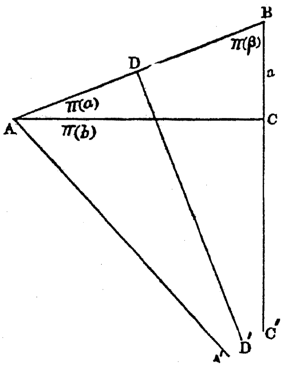

FIG. 32.

If from B we lay off β on the hypothenuse c, then at the end point D, (Fig. 32), within the triangle erect upon AB the perpendicular DD', and from the point A parallel to DD' draw AA', so will BC with its prolongation CC' be the third parallel; then is, angle $CAA' = \Pi(b)$, $DAA' = \Pi(c-\beta)$, consequently $\Pi(c-\beta) = \Pi(a) + \Pi(b)$. The last equation is then also still valid, when $c = \beta$, or $c < \beta$.

If $c = \beta$ (Fig. 33), then the perpendicular AA' erected upon

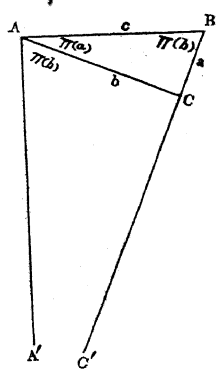

FIG. 33.

AB at the point A is parallel to the side $BC = a$, with its prolongation, CC', consequently we have $\Pi(a) + \Pi(b) = \frac{1}{2}\pi$, whilst also $\Pi(c-\beta) = \frac{1}{2}\pi$, (Theorem 23).

If $c < \beta$, then the end of β falls beyond the point A at D (Fig. 34) upon the prolongation of the hypothenuse AB. Here the perpendicular DD' erected upon AD, and the line

AA′ parallel to it from A, will likewise be parallel to the side BC$=a$, with its prolongation CC′.

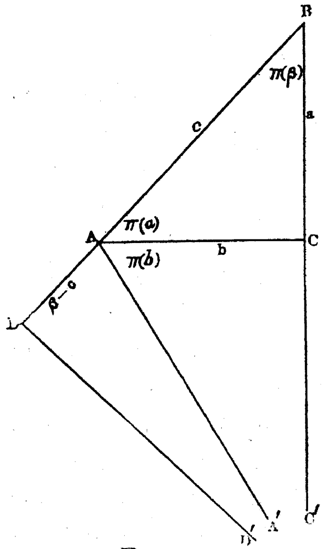

FIG. 34.

Here we have the angle DAA′ $=\Pi(\beta-c)$, consequently

$\Pi(a)+\Pi(b)=\pi-\Pi(\beta-c)=\Pi(c-\beta)$, (Theorem 23).

The combination of the two equations found gives,

$$2\Pi(b)=\Pi(c-\beta)+\Pi(c+\beta),$$
$$2\Pi(a)=\Pi(c-\beta)-\Pi(c+\beta),$$

whence follows

$$\frac{\cos\Pi(b)}{\cos\Pi(a)}=\frac{\cos[\frac{1}{2}\Pi(c-\beta)+\frac{1}{2}\Pi(c+\beta)]}{\cos[\frac{1}{2}\Pi(c-\beta)-\frac{1}{2}\Pi(c+\beta)]}$$

Substituting here the value, (Theorem 35)

$$\frac{\cos\Pi(b)}{\cos\Pi(a)}=\cos\Pi(c),$$

we have

$$[\tan\tfrac{1}{2}\Pi(c)]^2=\tan\tfrac{1}{2}\Pi(c-\beta)\tan\tfrac{1}{2}\Pi(c+\beta).$$

Since here β is an arbitrary number, as the angle $\Pi(\beta)$ at the one side of c may be chosen at will between the limits 0 and $\frac{1}{2}\pi$, consequently β between the limits 0 and ∞, so we may deduce by taking consecutively $\beta=c$, $2c$, $3c$, &c., that for every positive number n, $[\tan\frac{1}{2}\Pi(c)]^n=\tan\frac{1}{2}\Pi(nc)$.

If we consider n as the ratio of two lines x and c, and assume that $\cot\frac{1}{2}\Pi(c)=\mathbf{e}^c$,

then we find for every line x in general, whether it be positive or negative, $\tan\frac{1}{2}\Pi(x)=\mathbf{e}^{-x}$

where $\mathbf{e}$ may be any arbitrary number, which is greater than unity, since $\Pi(x)=0$ for $x=\infty$.

Since the unit by which the lines are measured is arbitrary, so we may by $\mathbf{e}$ also understand the base of the Napierian Logarithms.

37. Of the equations found above in Theorem 35 it is sufficient to know the two following,

$$\sin\Pi(c)=\sin\Pi(a)\,\sin\Pi(b),$$
$$\sin\Pi(a)=\sin\Pi(b)\cos\Pi(\beta),$$

applying the latter to both the sides a and b about the right angle, in order from the combination to deduce the remaining two of Theorem 35, without ambiguity of the algebraic sign, since here all angles are acute.

In a similar manner we attain the two equations

$$(1.)\quad \tan\Pi(c)=\sin\Pi(a)\tan\Pi(a),$$
$$(2.)\quad \cos\Pi(a)=\cos\Pi(c)\cos\Pi(\beta).$$

We will now consider a rectilineal triangle whose sides are a, b, c, (Fig. 35) and the opposite angles A, B, C.

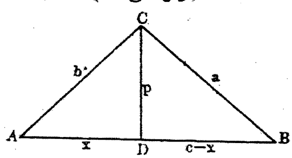

FIG. 35.

If A and B are acute angles, then the perpendicular p from the vertex of the angle C falls within the triangle and cuts the side c into two parts, x on the side of the angle A and $c-x$ on the side of the angle B. Thus arises two right-angled triangles, for which we obtain, by application of equation (1),

$$\tan\Pi(a)=\sin B\tan\Pi(p),$$
$$\tan\Pi(b)=\sin A\tan\Pi(p),$$

which equations remain unchanged also when one of the angles, *e. g.* B, is a right angle (Fig. 36) or an obtuse angle (Fig 37).

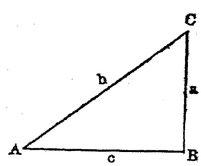

FIG. 36.

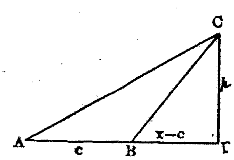

FIG. 37.

Therefore we have universally for every triangle

$$(3).\qquad \sin A\tan\Pi(a)=\sin B\tan\Pi(b).$$

For a triangle with acute angles A, B, (Fig. 35) we have also (Equation 2),

$$\cos \Pi(x) = \cos A \cos \Pi(b),$$
$$\cos \Pi(c-x) = \cos B \cos \Pi(a),$$

which equations also relate to triangles, in which one of the angles A or B is a right angle or an obtuse angle.

As example, for $B=\frac{1}{2}\pi$ (Fig. 36) we must take $x=c$, the first equation then goes over into that which we have found above as Equation 2, the other, however, is self-sufficing.

For $B > \frac{1}{2}\pi$ (Fig. 37) the first equation remains unchanged, instead of the second, however, we must write correspondingly

$$\cos \Pi(x-c) = \cos(\pi - B) \cos \Pi(a);$$

but we have $\cos \Pi(x-c) = -\cos \Pi(c-x)$

(Theorem 23), and also $\cos(\pi - B) = -\cos B$.

If A is a right or obtuse angle, then must $c-x$ and x be put for x and $c-x$, in order to carry back this case upon the preceding.

In order to eliminate x from both equations, we notice that (Theorem 36)

$$\cos \Pi(c-x) = \frac{1-[\tan \frac{1}{2} \Pi(c-x)]^2}{1+[\tan \frac{1}{2} \Pi(c-x)]^2}$$
$$= \frac{1-e^{2x-2c}}{1+e^{2x-2c}} =$$
$$= \frac{1-[\tan \frac{1}{2} \Pi(c)]^2 [\cot \frac{1}{2} \Pi(x)]^2}{1+[\tan \frac{1}{2} \Pi(c)]^2 [\cot \frac{1}{2} \Pi(x)]^2}$$
$$= \frac{\cos \Pi(c) - \cos \Pi(x)}{1 - \cos \Pi(c) \cos \Pi(x)}$$

If we substitute here the expression for $\cos \Pi(x)$, $\cos \Pi(c-x)$, we obtain

$$\cos \Pi(c) = \frac{\cos \Pi(a) \cos B + \cos \Pi(b) \cos A}{1 + \cos \Pi(a) \cos \Pi(b) \cos A \cos B}$$

whence follows

$$\cos \Pi(a) \cos B = \frac{\cos \Pi(c) - \cos A \cos \Pi(b)}{1 - \cos A \cos \Pi(b) \cos \Pi(c)}$$

and finally

$$[\sin\Pi(c)]^2=[1-\cos B\cos\Pi(c)\cos\Pi(a)][1-\cos A\cos\Pi(b)\cos\Pi(c)]$$

In the same way we must also have

(4.)

$$[\sin\Pi(a)]^2=[1-\cos C\cos\Pi(a)\cos\Pi(b)][1-\cos B\cos\Pi(c)\cos\Pi(a)]$$

$$[\sin\Pi(\mathrm{b})]^2=[1-\cos A\cos\Pi(b)\cos\Pi(c)][1-\cos C\cos\Pi(a)\cos\Pi(b)]$$

From these three equations we find

$$\frac{[\sin\Pi(b)]^2]\sin\Pi(c)]^2}{[\sin\Pi(a)]^2}=[1-\cos A\cos\Pi(b)\cos\Pi(c)]^2.$$

Hence follows without ambiguity of sign,

$$(5.)\qquad \cos A\cos\Pi(b)\cos(c)+\frac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)}=1.$$

If we substitute here the value of $\sin\Pi(c)$ corresponding to equation (3.)

$$\sin\Pi(c)=\frac{\sin A}{\sin C}\tan\Pi(a)\cos\Pi(c)$$

then we obtain

$$\cos\Pi(c)=\frac{\cos\Pi(a)\sin C}{\sin A\sin\Pi(b)+\cos A\sin C\cos\Pi(a)\cos\Pi(b)};$$

but by substituting this expression for $\cos\Pi(c)$ in equation (4),

$$(6.)\qquad \cot A\ \sin C\ \sin\Pi(b)+\cos C=\frac{\cos\Pi(b)}{\cos\Pi(a)}.$$

By elimination of $\sin\Pi(b)$ with help of the equation (3) comes

$$\frac{\cos\Pi(a)}{\cos\Pi(b)}\cos C=1-\frac{\cos A}{\sin B}\sin C\ \sin\Pi(a).$$

In the meantime the equation (6) gives by changing the letters,

$$\frac{\cos\Pi(a)}{\cos\Pi(b)}=\cot B\ \sin C\ \sin\Pi(a)+\cos C.$$

From the last two equations follows,

$$(7.)\qquad \cos A+\cos B\ \cos C=\frac{\sin B\ \sin C}{\sin\Pi(a)}$$

All four equations for the interdependence of the sides a, b, c, and the opposite angles A, B, C, in the rectilineal triangle will therefore be, [Equations (3), (5), (6), (7),]

$$
(8.)\quad \begin{cases}
\sin A \tan \Pi(a) = \sin B \tan \Pi(b), \\
\cos A \cos \Pi(b) \cos \Pi(c) + \dfrac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)} = 1, \\
\cot A \ \sin C \sin \Pi(b) + \cos C = \dfrac{\cos \Pi(b)}{\cos \Pi(a)}, \\
\cos A + \cos B \cos C = \dfrac{\sin B \ \sin C.}{\sin \Pi(a)}
\end{cases}
$$

If the sides a, b, c of the triangle are very small, we may content ourselves with the approximate determinations (Theorem 36)

$$
\begin{aligned}
&\cot \Pi(a) = a, \\
&\sin \Pi(a) = 1 - \tfrac{1}{2}a^2 \\
&\cos \Pi(a) = a,
\end{aligned}
$$

and in like manner also for the other sides b and c.

The equations 8 pass over for such triangles into the following,

$$
\begin{aligned}
&b \sin A = a \sin B, \\
&a^2 = b^2 + c^2 - 2bc \cos A, \\
&a \sin(A + C) = b \sin A, \\
&\cos A + \cos(B + C) = 0.
\end{aligned}
$$

Of these equations the first two are assumed in the ordinary geometry; the last two lead, with help of the first, to the conclusion $A + B + C = \pi$.

Therefore the imaginary geometry passes over into the ordinary, when we suppose that the sides of a rectilineal triangle are very small.

I have, in the scientific bulletins of the University of Kasan, published certain researches in regard to the measurement of curved lines, of plane figures, of the surfaces and the volumes of solids, as well as in relation to the application of imaginary geometry to analysis.

The equations (8.) attain for themselves already a sufficient foundation for considering the assumption of imaginary geometry as possible. Hence there is no means, other than astronomical observations, to use for judging of the exactitude which pertains to the calculations of the ordinary geometry.

This exactitude is very far-reaching, as I have shown in one of my investigations, so that, for example, in triangles whose sides are attainable for our measurement, the sum of the three angles is not indeed different from two right angles by the hundredth part of a second.

In addition, it is worthy of notice, that the four equations (8.) of plane geometry pass over into the equations for spherical triangles, if we put $a\sqrt{-1}, b\sqrt{-1}, c\sqrt{-1}$ instead of the sides a, b, c; with this change however, we must also put

$$\sin \Pi(a) = \frac{1}{\cos(a)},$$

$$\cos \Pi(a) = (\sqrt{-1}) \tan a,$$

$$\tan \Pi(a) = \frac{1}{\sin a\,(\sqrt{-1})},$$

and similarly also for the sides b and c.

In this manner we pass over from equations (8) to the following,

$$\sin A \sin b = \sin B \sin a,$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cot A \sin C + \cos C \cos b = \sin b \cot a,$$

$$\cos A = \cos a \sin B \sin C - \cos B \cos C.$$

AN ELEMENTARY DEMONSTRATION OF THE EXPANSION OF THE SINE AND COSINE IN TERMS OF THE CIRCULAR MEASURE.

By Prof. W. H. Echols, Rolla, Mo.

The rigorous *deduction* of these series and the proof of the possibility of the expansion seems only possible through aid of the calculus or the complex quantity in the shape of De Moivre's Theorem. Their great importance to the engineer and elementary student has, nevertheless, caused more elementary demonstrations to be desired. An example of such is seen in Prof. Newcomb's Trigonometry, 1889. Of this class the following seems quite simple and direct. The method of deriving the coefficients, suggested by Mr. Schaeberle's Demonstration of the Logarithmic Series in Annals of Mathematics, is not very different from that employed by Dr. W. B. Smith in his Clue to Trigonometry, just out of press

I.

Assume this expansion

$$(1.) \qquad \sin x+\cos x=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx_n+.$$

$$(2.) \qquad \therefore-\sin x+\cos x=a_0-a_1x+a_2x^2-\ldots\ldots(-)^na_nx^n+.$$

Adding and subtracting

$$(3.) \qquad \sin x=a_1x+a_3x^3+\ldots\ldots+a_{2n-1}x^{2n-1}+.$$

$$(4.) \qquad \cos x=a_0+a_2x^2+\ldots\ldots+a_{2n-2}x^{2n-2}+.$$

$x=0$ in (1) or (4) gives $a_0=1$. Divide (3) by x, then $x=0$ gives $a_1=1$.

Square (1) and save only the terms in x^{odd} because of (3).

(5.) $\therefore \sin 2x = 2x + 2(a_3 + a_2)x^3 + 2(a_5 + a_4 + a_2 a_3)x^5 + \ldots$

(1) $\times$ (2) gives

(6.) $\cos 2x = 1 + (2a_2 - 1)x^2 + 2a_4 - 2a_3 + a_2^2)x_4 + \ldots$

Put $2x$ for x in (1) and equate to the sum of (5) and (6).

Thus

$$\sin 2x + \cos 2x = \left|\begin{array}{l} 1 \\ +2x \\ +(2a_2 - 1)x^2 \\ +2(a_3 + a_2)x^3 \\ +(2a_4 - 2a_3 + a_2^2)x^4 \\ \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}\right. = \left|\begin{array}{l} 1 \\ +2x \\ +2^2 a_2 x^2 \\ +2^3 a_3 x^3 \\ +2^4 a_4 x^4 \\ \quad \cdot\ \cdot\ \cdot \end{array}\right.$$

Identifying coefficients of like powers of x we have

$$a_0 = +1,\ a_1 = +1,\ a_2 = -\frac{1}{2!},\ a_3 = -\frac{1}{3!},\ a_4 = +\frac{1}{4!},\ a_5 = +\frac{1}{5!},\ \text{etc.}$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{2n-1}}{2^{2n-1}} \sum_{m=0}^{m=2n-1} a_m a_{2n-1-m} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n-2}}{2^{2n-2}} \sum_{m=0}^{m=2n-2} (-)^m a_m a_{2n-2-m} + \ldots$$

II.

In forming the series of products $\Sigma a_p a_q$ involved in the coefficients above we observe the law of their signs to be such that when p and q are each odd the product is negative, and otherwise positive.

Assuming the law of the coefficients as found above to hold good to the nth term inclusive in each series; we notice,

1° The law for the signs of the coefficients

$$+a_1,\ -a_3,\ +a_5,\ -a_7,\ \ldots\ (-)^{n+1} a_{2n-1},$$
$$+a_0,\ -a_2,\ +a_4,\ -a_6,\ \ldots\ (-)^{n+1} a_{2n-2},$$

when applied to the expressions

$$a_0a_{2n-1}+a_1a_{2n-2}+\ldots+a_1a_{2n-2}+a_0a_{2n-1},$$

$$a_0a_{2n-2}+a_1a_{2n-3}+\ldots+a_1a_{2n-3}+a_0a_{2n-2},$$

is such that the sign of each is $(-)^{n+1}$, and this is therefore the sign of the nth term in each series.

2° In finding the numerical value of the coefficient of the nth term in either series we need only consider the expression

$$\sum_{m=0}^{m=n}\frac{1}{m!(n-m)!}=\frac{1}{m!}[1+y_{=1}]^m=\frac{2^m}{m!}$$

Therefore the coefficient a_r in either series is determined by

$$a_r x^r=\frac{2^r}{r!}\,\frac{x^4}{2^r}$$

$$\therefore\ a_r=\frac{1}{r!}$$

The nth terms of the series are

$$(-)^{n+1}\frac{x^{2n-1}}{(2n-1)!}\text{ and }(-)^{n+1}\frac{x^{2n-2}}{(2n-2)!}$$

III

We are to prove that the law of the coefficients assumed to the nth term inclusive is also true for the $(n+1)$th term.

In the series for the sine we have for the coefficient of the $(n+1)$th term

$$a_{2n+1}=\frac{1}{2^{2n+1}}\sum_{m=0}^{m=2n+1}a_m a_{2n+1-m},$$

or

$$2^{2n+1}a_{2n+1}=2a_{2n+1}(-)^n\frac{2}{(2n+1)!}(-)^{n+1}\frac{1}{(2n+1)!}\sum_{m=0}^{m=2n+1}\frac{1}{m!(2n+1-m)!}$$

$$(2^{2n+1}-2)a_{2n+1}=(-)^{n+1}\frac{2^{2n+1}-2}{(2n+1)!}$$

$$\therefore\ a_{2n+1}=(-)^{n+1}\frac{1}{(2n+1)!}$$

In like manner, in the cosine series,

$$2^{2n}a_{2n}=2a_{2n}(-)^{n+1}\sum_{m=1}^{m=2n-1}{}' a_m a_{2n-m} ,$$

$$=2a_{2n}(-)^n \frac{2}{(2n)!}(-)^{n+1}\sum_{0}^{2n}{}' \frac{1}{m!(2n-m)!} ,$$

$$\therefore a_{2n}=(-)^{n+1}\frac{1}{(2n)!} .$$

The law of the series being proved to the third term, it is true for the fourth and all succeeding terms; since it satisfies the test of convergency the first assumption was justifiable.

Thus, if the series be possible, we have

$$\sin x=x-\ldots(-)^{n+1}\frac{x^{2n+1}}{(2n+1)!}+$$

$$\cos x=1-\ldots(-)^{n+1}\frac{x^{2n-2}}{(2n-2)!}+.$$

THE TWO-TERM PRISMOIDAL FORMULA.

BY DR. GEORGE BRUCE HALSTED, UNIVERSITY OF TEXAS.

I

Prof. Echols begins an interesting article on the Volume of the Prismoid as follows: "In estimating the volume of earthwork in the construction of lines of communication, a particular solid has occurred so frequently that engineers have given it a specific name; the Prismoid. Whether the word was used to designate a definite geometrical solid prior to its adoption by engineers for that purpose, I have been unable to discover. The solid has been an extremely interesting one to engineers, and much has been written by them upon the subject of its volume." Is it not surprising, then, that they have not found out what the world has possessed for more than a decade—a Two-Term Prismoidal Formula?

The word Prismoid is a good, old, mathematical term, and has always had and kept, and I hope may always keep, the meaning recognized, for example, by Charles Hutton in his Mathematical Dictionary, the new edition of which was published in London in 1815. There, under the word, you read as follows: "Prismoid . . . Its ends are any dissimilar parallel plane figures of the same number of sides; the upright sides being trapezoids. If the ends of the prismoid be bounded by dissimilar curves, it is sometimes called a cylindroid." This meaning the word maintained down to my own college days at Princeton, where I remember it in the text-books of Loomis, and still maintains, see for example the word *Mensuration*, in the latest edition of the Encyclopædia Britannica.

But the formula called by the name of this solid, and henceforth to be called the old or three-term Prismoidal formula, went far beyond the prismoid in its exact applicability.

Newton (Methodus Differentialis, published 1711; further carried out by Cotes, on Newton's Meth. Dif. in the works collected posthumously, 1722) showed how an area or volume could be evaluated approximately from parallel cross-sections, and especially that from three cross-sections, following at the same distance apart, we get approximately the enclosed segment if we add the outer sections to four times the mid section and multiply the sum by a sixth of the distance between the outer sections.

Maclaurin (1742, Fluxions, No. 848) referring both to Newton and Cotes, made additions which indicate that this special rule of Newton's, the Old Prismoidal Formula, gives the content exactly when every section parallel to the base is a function of its distance from it of a degree not higher than the third, $\omega(x)=n_0+n_1x+n_2x^2+n_3x^3$.

After a century of applications to areas and volumes, in 1842 Steiner conquered it by elementary geometry and indicated its applicability to warped or ruled surfaces.

But, in seeming ignorance of all this, American engineers began and continue to give their time to doing over again what had been already done.

In his Field-Book, Edition 1854, Henck says. "A prismoid is a solid having two parallel faces, and composed of prisms, wedges and pyramids, whose common altitude is the perpendicular distance between the parallel faces." This is ambiguous and stupid. It defines nothing. The old prismoid may be cut up into prisms, wedges, and pyramids, but this is not at all its essence, and, to me, does not even definitely suggest the more general solid for which in 1881 I introduced to English readers the word Prismatoid, now adopted by the Encyclopædia Britannica, which solid is defined by Prof. Echols as "having two parallel plane polygons for bases, and whose side surface is made up of plane faces (triangles or quadrilaterals) formed by joining corresponding corners of the bases. Using *corresponding* corners to denote any two corners, one of each base, such that the straight line joining them is an *edge* of the" prismatoid.

I think this definition by Prof. Echols is faulty, because it directs you to join corresponding corners, when in reality there are no corresponding corners, as his second sentence discloses in telling you that corresponding corners denote such as you have made corresponding corners by joining corresponding corners.

I venture to suggest as better my own definition given more than ten years ago, and appearing in four successive editions of my Mensuration (Ginn & Co.) and four successive editions of my Geometry (Wiley & Sons), as follows:

A Prismatoid is a polyhedron whose bases are any two polygons in parallel planes, and whose lateral faces are triangles determined by so joining the vertices of these bases that each lateral edge, with the preceding, forms a triangle with one side of either base.

Yet the mis-named prismoidal formula corresponded in range neither with the prismoid, nor the prismatoid, nor their limiting form, the cylindroid. Maclaurin had indicated exactly its applicability. Yet, in 1857, fifteen years after Steiner, Gillespie reaped honor from merely showing that the formula is applicable to the space covered by the hyperbolic-paraboloid.

In 1858 Chauncey Wright in the Mathematical Monthly (Cambridge, Mass.,) in a special investigation devoted to the subject, obtained by the Differential Calculus (which was not at all necessary) the cubic equation of applicability, but missed Weddle's beautiful rule.

Prof. E. W. Hyde, in 1876, in an article entitled Limits of the Prismoidal Formula, did not even get as far as his predecessors.

In an extended memoir on the Prismoidal Formula, in Van Nostrand's Magazine 1879, J. W. Davis, again by the differential calculus, reaches the cubic criterion.

At that date my own Mensuration was already written, and I was teaching it regularly to my classes. In it the applicability of the old or three-term prismoidal formula was exhaustively treated and without the calculus. For readers of the calculus the following may be given as a paraphrase of the method quoted by Prof. Echols from Todhunter Int. Cal., p. 173, of showing that this formula applies exactly to all solids whose cross-sections are cubic functions of the section-height.

If $A_x = \omega(x) = n_0 + n_1 x + n_2 x^2 + n_3 x^3$,

then $\omega(o) + 4\omega(\frac{1}{2}a) + \omega(a) = n_0$

$$
\begin{aligned}
&+ 4n_0 + 2an_1 + a^2 n_2 + \tfrac{1}{2}a^3 n_3 \\
&+ \ n_0 + \ an_1 + a^2 n_2 + \ a^3 n_3 \\
&= 6n_0 + 3an_1 + 2a^2 n_2 + \tfrac{3}{2}a^3 n_3.
\end{aligned}
$$

Thus $D = \frac{1}{6}a[B_1 + 4M + B_2] = \frac{1}{6}a[\omega(o) + 4\omega(\frac{1}{2}a) + \omega(a)]$

$$= \tfrac{1}{6}a[6n_0 + 3an_1 + 2a^2 n_2 + \tfrac{3}{2}a^3 n_3] = an_0 + \tfrac{1}{2}a^2 n_1 + \tfrac{1}{3}a^3 n_2 + \tfrac{1}{4}a^4 n_3.$$

But by the calculus this is the exact volume of the solid, since it is $= \int_0^a \omega(x)$.

This investigation is faulty and does not fix the criterion of applicability, since it says nothing to show that the conditions are satisfied only by functions which have no fourth or higher powers. This is proved in my Menstration without the calculus. For readers of the calculus the following method may be of interest.

Measuring x on a line normal to which the sections are made, let $A_x = \omega(x)$ be the area of the section at the distance x from the origin. Let three sections be made through any solid at the distance $(x-h)$, the distance x, and the distance $(x+h)$ from the origin. Then $\omega(x-h)$, $\omega(x)$, $\omega(x+h)$ will be the areas of these sections, and the old Prismoidal Formula, for the volume between the bases $\omega(x-h)$ and $\omega(x+h)$ gives

$$\tfrac{1}{3}h[\omega(x-h)+4\omega(x)+\omega(x+h)].$$

But the volume is the integral of the differential solid $\omega(x)dx$ between the limits $x-h$ and $x+h$.

$$\int_{x-h}^{x+h}\omega(x)dx=\int\omega(x+h)dx-\int\omega(x-h)dx.$$

If the function ω fulfills the conditions of the Prismoidal Formula, we have, by equating the two expressions for the volume,

$$\int\omega(x+h)dx-\int\omega(x-h)dx=\tfrac{1}{3}h[\omega(x-h)+4\omega(x)+\omega(x+h)].$$

To find what form of ω will satisfy the equation, develop both its members by Taylor's Theorem.

The first member becomes (A)

$$\int[\omega(x)+\omega'(x)h+\omega''(x)\frac{h^2}{2}+\omega'''(x)\frac{h^3}{2\times 3}+\text{etc.}]dx$$

$$-\int[\omega(x)-\omega'(x)h+\omega''(x)\frac{h^2}{2}-\omega'''(x)\frac{h^3}{2.3}+\text{etc.}]dx$$

$$=\int[2\omega'(x)hdx+2\omega'''(x)\frac{h^3}{2.3}dx+\text{etc.}]$$

$$=2\omega(x)h+\omega''(x)\frac{h^3}{3}+\omega''''(x)\frac{h^5}{3.4.5}+\text{etc.}$$

The second member becomes (B)

$$\tfrac{1}{3}h[\omega(x)-\omega'(x)h+\omega''(x)\frac{h^2}{2}-\text{etc.}+4\omega(x)+\omega(x)+\omega'(x)h+\omega''(x)h^2+\text{etc.}]$$

$$=\tfrac{1}{3}h[6\omega(x)+\omega''(x)h^2+\omega''''(x)\frac{h^4}{3.4}+\text{etc.}$$

$$=2\omega(x)h+\omega''(x)\frac{h^3}{3}+\omega''''(x)\frac{h^5}{36}+\text{etc.}$$

Comparing the last members of (A) and (B), we find the equation,

$$2\omega(x)h+\omega''(x)\frac{h^3}{3}+\omega''''(x)\frac{h^5}{60}+\text{etc.}=$$

$$=2\omega(x)h+\omega''(x)\frac{h^3}{3}+\omega''''(x)\frac{h^5}{36}+\text{etc.}$$

Therefore the old Prismoidal Formula applies exactly to *all solids* contained between two parallel planes, of which the area of any section parallel to these planes can be expressed by a rational integral algebraic function, of a degree not higher than the third, of its distance from either of these bounding planes or bases. And in general it applies universally to no other solids.

Thus the cubic $A_x = n_0 + n_1 x + n_2 x^2 + n_3 x^3$

expresses the law of variation in magnitude of the plane generatrix of prismoidal spaces.

But our prismatoid needs only a quadratic. This is readily proved. Any prismatoid may be divided into tetrahedra, all of the same altitude as the prismatoid; some having their apex in the upper base of the prismatoid, and for base a portion of its lower base; some having base in the upper, and apex in the lower base of the prismatoid; and others having for a pair of opposite edges a sect in the plane of each base of the prismatoid. A section A_x of a tetrahedron in the first position equals

$$\frac{(a-x)^2 B_1}{a^2}.$$

For the second position $A_x = \dfrac{x^2 B_2}{a^2}$.

For the third position $A_x = \pi(ax - x^2)$,

(see Halsted's Geometry, page 250.

Thus in any prismatoid any cross-section is only a quadratic function of its distance from either base. Therefore in employing for its volume the old three-term formula, we have been using a bear-trap to catch a mouse.

For all solids whose section is a function of degree not higher than the second, or

$$A_x = \omega(x) = n_0 + n_1 x + n_2 x^2,$$

the volume is $\int_0^a \omega(x) = n_0 a + \frac{1}{2} n_1 a^2 + \frac{1}{3} n_2 a^3$.

Measuring x from one base B_1, we have

$$A_0=B_1=n_0.$$

Then $$A_a=B_2=B_1+n_1a+n_2a^2.$$

We see at once that any cross-section whatever, if known in addition to the altitude and bases, will give us the volume.

Suppose we know the section at $1/z$ the height of the solid above B_1, then we have for determining n_1 and n_2 the two equations

$$A_{\frac{a}{z}}=B_1+n_1\frac{a}{z}+n_2\frac{a^2}{z^2}$$

$$B_2=B_1+n_1a+n_2a^2.$$

Hence $$n_1=\frac{z^2A_{\frac{a}{z}}-(z^2-1)B_1-B_2}{(z-1)a},$$

$$n_2=\frac{zB_2+z(z-1)B_1-z^2A_{\frac{a}{z}}}{(z-1)a^2}.$$

Thus for the volume of the solid we have

$$V=\frac{a}{6(z-1)}[(2z-3)B_2-(z-1)(z-3)B_1+z^2A_{\frac{a}{z}}$$

Stretching two hands at us from this, we see a two-term prismoidal formula.

For $z=3$, this gives

$$V=\tfrac{1}{4}a(B_2+3A_{\frac{a}{3}}).$$

Again, for $2z=3$.

$$V=\tfrac{1}{4}a(B_1+3A_{\frac{2}{3}a}).$$

Hence, to find the volume of a prismatoid, or of any solid whose section gives a quadratic:

Multiply one-fourth its altitude by the sum of one base and three times a cross-section at two-thirds the altitude from that base.

$$D=\tfrac{1}{4}a(B+3T).$$

II

The rule to find the volume of a prismoid, a prismatoid, or any solid whose section is expressable as a quadratic.

Multiply one-fourth its altitude by the sum of one base and three times a cross-section at two-thirds the altitude from that base.

Proved without the calculus in my Mensuration, where the formula is written

$$V=\tfrac{1}{4}a(B_2+3A_{\frac{a}{3}}).$$

and also

$$V=\tfrac{1}{4}a(B_1+3A_{\frac{2}{3}a}).$$

Proved synthetically in my Geometry, where the formula is written

$$D=\tfrac{1}{4}a(B+3T),$$

is in my estimation incomparably simpler than every other; but if we are willing to make and use an auxiliary solid, we may by its help express otherwise in two terms our prismoidal volume.

The already-mentioned article by Prof. Echols gives prominence to a three term formula using such an auxiliary, "which the writer first heard enunciated by Prof. W. M. Thornton, of the University of Virginia, about ten years ago, but which he has never seen in print."

It was published about a decade ago in my Mensuration in the following form: Twice the volume of the segment of a ruled surface between parallel planes is equivalent to the sum of the cylinders on its bases, diminished by the cone whose vertex is in one of the parallel planes, and whose elements are respectively parallel to the lines of the ruled surface.

This theorem is nearly a hundred years old, as also the name "associate cone" for the cone it defines and uses, and the designation "cylindroid" for such a segment. This with analogous cubatures was given in 1806 by Meier Hirsh, well known by his table of integrals. It was geometrically obtained and so interpreted by Koppe in 1838 (Crelle's Journal, 18 p. 275). It afterwards appeared in his "Neuer Lehrsatz der Stereometrie," Essen, 1843. (Compare Grunerts Archiv, 9 p. 82). We may build a demonstration of it synthetically on the simpler proposition given by Tinseau in 1780 and re-stated in the Mensuration as follows: A solid is bounded by the triangles ABC, CBD, the parallelogram ACDE, and the skew quadrilateral BAED, whose elements are parallel to the plane BCD. Find its volume. Answer $\frac{1}{2}a$. ABC. For on completing the prism with the parallel edges CD, AE, BF, the elements of the skew quadrilateral appear as

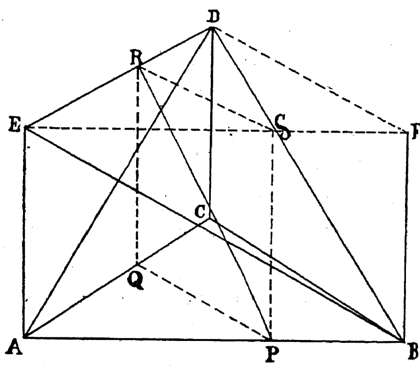

FIG. 38.

diagonals of parallelograms, as PQRS, in which the prism is cut by planes parallel to BCDF; hence the hyperbolic paraboloid halves the prism of height a and base ABC. Similarly we see that a tetrahedron is bisected by the hyperbolic paraboloid whose directrices are two opposite edges, and whose plane directer is parallel to another pair of opposite edges, a theorem first given by Möbius on page 238 of his celebrated "Barycentrische Calcul," Leipzig, 1827, in the following form: "Construct on the surface of a hyperbolic paraboloid a rectilineal quadrilateral, then will the pyramid whose summits are the vertices of the quadrilateral be halved by the surface." The next step may be thus individualized: A solid is bounded by a parallelogram, two skew quadrilaterals, and two parallel triangles; find its volume. Answer, $\frac{1}{2}a(\triangle_1+\triangle_2)$. Let ABC, DEF be the triangles, ACDE the $\square$, CBFD, BAEF the skew

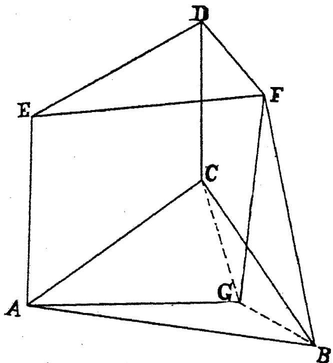

FIG. 39.

quadrilaterals. Complete the prism whose parallel edges are CD, AE, GF, altitude a, then from what precedes, the required volume is

$$\tfrac{1}{2}a.\text{ABG}+\tfrac{1}{2}a.\text{BCG}+\tfrac{1}{2}a\text{CAG}+\tfrac{1}{2}a\text{DEF}$$
$$=\tfrac{1}{2}a.\ (\triangle_1+\triangle_2).$$

This empowers us to deal with a solid bounded by two parallel triangles and three skew quadrilaterals, and introduces the formula $V=a.[\frac{1}{2}(B''+B')-\frac{1}{6}B_c]$.

But every cylindroid or prismatoid is made up of finite or indefinitely small solids like this. "This then is the rational formula for computing the volume of any cylindroid or" prismatoid, (Prof. Echols). But a great simplification of this into a real two-term prismoidal formula has long been known.

Prof. Echols gets by the calculus, the mid section

$$M=\tfrac{1}{2}(B''+B')-\tfrac{1}{4}B_c,$$

a result obtained geometrically by Steiner in 1842.

Substitute this and

$$V=a(M+\tfrac{1}{12}B_c),$$

the first two-term prismoidal formula, having been given by Koppe in his work already cited.

That *a cylindroid or prismatoid equals the cylinder or prism on its mid-section plus one-fourth the associate cone or pyramid*, should certainly be known to every engineer, for, as Prof. Echols suggests in our correspondence since his article, perhaps this two-term formula may involve less numerical work than even its more elegant younger sister

$$D=\tfrac{1}{4}a(B+3T).$$

COPY MULTIPLICATION TABLE,

By Mr. Levi W. Meech, Norwich, Conn.

Illustrations—The nature and use of this Copy Table may be first shown by comparing its results with the process by common arithmetic. Thus in *Example* **1**, the two tabular results copied at locations **OT** and **7A**, when read obliquely upward give 21, 07, 28, 35, 63, 14, which are the common product of each figure of the multiplicand by the multiplier 7, and their sum 2,202,144 is the total product.

Again in *Example* **2**, the line copied from location **7A** gives the unit figures only of the common product by 7 precisely as before. The next line copied from **8T** gives the *sum* of tens in the product by 7 added to the units of the common product by 8, that is 202361 added to 482026; this sum is 684387 as given by the Copy Table. And the next line copied from **OZ** gives the tens figures only of the common product by 8. The total sum 27369504 is evidently the true product sought.

So in *Example* **3**, the figures copied from locations **7A** and **8T** are identically the same as before. The next line copied from **4Z** is the similar sum of tens in the product by 8 added to units in the common product by 4, that is 203471 added to 246068, of which the tabular sum is 4494(1)39. It is important here, and in other similar instances, to note in illustration only, that when the sum of two such digits is 10 or more, as here 7+6 is (1)3, by the peculiar construction of the Copy Table, the (1) is carried to the next line below, and so included. Thus this (1) is here included by the tabular routine, in the last line for **ON**, giving 101240, instead of 101230, the simple

Advance figure of Multiplier located. First Entry.

		A	B	C	D	E	
M	012345	6789	6789	6789	6789	6789	
0	000000	0000	0001	0011	0111	1111	A
1	012345	6789					A
,			6780	6790	6890	7890	B
2	024680	2468	2469	2479	2579	3579	F
3	036925	8147	8148	8158	8258	9258	I
4	048260	4826	4827	4837	4937	5937	L
	012345	6789	6789	6789	6789	6789	
5	050505	0505	0506	0516	0616	1516	O
6	062840	6284	6285	6295	6395	7395	R
7	074185	2963	2964	2974			T
,					2074	3074	U
8	086420	8642	8643	8653	8753	9753	X
9	098765	4321	4322	4332	4432	5432	B′

Letter to locate with the next figure of Multiplier.

		J	K	L	M	N	
	012346	5789	5789	5789	5789	5789	
0	000112	1223	1233	2233	2333	2334	A
1	012458	6902	6912	7912			C
,					7012	7013	D
2	024794	1681	1691	2691	2791	2792	G
3	036030	6360	6370	7370	7470	7471	J
4	048376	1049	1059	2059	2159		M
,						2150	N
	012346	5789	5789	5789	5789	5789	
5	050612	6728	6738	7738	7838	7839	O
6	062958	1407	1417	2417	2517	2518	S
7	074294	6186	6196	7196	7296	7297	U
8	086530	1865	1875	2875	2976	2976	Y
9	098876	6544	6554	7554	7654	7655	B′

		T	U	V	W	
	012356	4789	4789	4789	4789	
0	001234	2456	2556	3566	3567	A
1	013580	6135	6235	7245	7246	E
2	025836	0814	0914	1924	1925	H
3	037182	4593	4693			J
,				5603	5604	K
4	049438	8272	8372	9381	9381	N
	012356	4789	4789	4789	4789	
5	051784	2951				P
,			2051	3061	3062	Q
6	063030	6630	6730	7740	7741	T
7	075386	0319	0419	1429		V
,					1420	W
8	087632	4098	4198			Z
,				5108	5109	A′
9	099988	8777	8877	9887	9888	B′

Letter to locate with the next figure of Multiplier.

		F	G	H	I	
	012356	4789	4789	4789	4789	
0	000011	0111	0112	1122	1222	A
1	012367	4890	4891			B
,				5801	5901	C
2	024613	8579				F
,			8570	9580	9680	G
3	036969	2258	2259	2269	3369	I
4	048215	6937	6938	7948		L
,					7048	M
	012356	4789	4789	4789	4789	
5	050561	0616	0617	1627	1727	P
6	062817	4395	4396			R
,				5306	5406	S
7	074163	8074	8075	9085	9185	U
8	086419	2753	2754	3764	3864	X
9	098765	6432	6433	7443	7543	B′

		O	P	Q	R	S	
	012346	5789	5789	5789	5789	5789	
0	001123	2344	2345	2445	3445	3455	A
1	013469	7023	7024	7124	8124	8134	D
2	025705	2702	2703	2803	3803	3813	H
3	037041	7481	7482	7582	8582	8592	J
4	049387	2160	2161	2261	3261	3271	N
	012346	5789	5789	5789	5789	5789	
5	051623	7849					O
,			7840	7940	8940	8950	P
6	063969	2528	2529	2629	3629	3639	S
7	075205	7207	7208	7308	8308	8318	V
8	087541	2986	2987				Y
,				2087	3087	3097	Z
9	099887	7965	7666	7866	8766	8776	B′

		X	Y	Z	A′	B′	
	012345	6789	6789	6789	6789	6789	
0	001234	4567	5567	5667	5677	5678	A
1	013579	0246	1246	1346	1356	1357	E
2	025814	6925	7925				H
,				7025	7035	7036	I
3	037159	2604	3604	3704	3714	3715	K
4	049494	8383	9383	9483	9493	9494	N
	012345	6789	6789	6789	6789	6789	
5	051739	4062	5062	5162	5172	5173	Q
6	063074	0741	1741	1841	1851	1852	T
7	075819	6420	7420	7520	7530	7531	W
8	087654	2109	3109	3209	3219		A′
,						3210	B′
9	099999	8888	9888	9988	9998	9999	B′

tens in the product by 4. The sum of the regular tabular elements gives the true product 153206304.

Example **1.**

```
 314592×⁰7
 178534 7A
202361  OT
2202144 loc.
```

Example **2.**

```
 314592×⁰87
 178534 7A
684387  8T
203471  OZ
27369504 loc.
```

Example **3.**

```
  314592×⁰487
  178534 7A
 684387  8T
448439   4Z
101240   ON
153206304 loc.
```

Example **4.**

```
   314592×⁰3487
   178534 7A
  684387  8T
 449439   4Z
033716    3N
101130    OJ
1096982304 loc.
```

USE OF THE TABLE: Prefix a cypher to the multiplier, which will then indicate the number of locations of the left pointer. At each location we copy as many figures in the product as there are digits in the Multiplicand; and their sum gives the true product. Thus in *Example* **2**, the right-hand figure of the Multiplier being 7, we locate (First entry) the left pointer at **7A**, or the junction of **7** at the side with **A** at the top of the upper left portion of the Table. Looking up to the nearest heavy line above, we find just under it, 3 1 the two left digits of the Multiplicand, and directly under those are 1 and 7 to be copied in the product with the right hand, while the left pointer stays at the location. Next, under 4 5 from the Multiplicand are 8 5 at the pointer to be copied in the product. Then under 9 2 from the Multiplicand are 3 4 at the pointer to be copied in the product, which completes the six figures. Now **T** being on the right of the first location, and 8 the next left figure of the Multiplier, let the pointer be moved to *the second* *location* **8T**, or **8** at the side and **T** above. Here just under the next heavy line above find as before 3 1 from the Multiplicand under which at the pointer are 6 8 to be copied with the right hand in the product, beginning one place further to the left. Next 4 5 and 9 2 from the Multiplicand give successively on the line of the pointer, 4 3 and 8 7 for the product. Lastly, with **O** from the Multiplier, and **Z** from the right of the pointer, move to *the third location* **OZ**, and proceed with the Multiplicand, as before, and as if the left column headed 012345 were united to column Z. And thus the simple mode of copying shown in the wrought Examples is entirely general, however large may be the two factors to be multiplied.

It may be proper to state that the present Copy Table is condensed from the full Table inserted on pages 62, 63 of the "System and Tables of Life Insurance," or Experience of the Thirty American Offices. Besides illustrating how "the carrying figure" of the schools can be replaced by a simple routine, this Table has opened the way for another improved Copy Table, a volume now in preparation.

THE TRANSIT OF MERCURY ACROSS THE SUN'S DISK ON MAY 9th, 1891.

BY MR. H. C. WILLIAMS. COLUMBIA, MO.

The mathematics used in computing the phases of a transit of a minor planet across the sun's disk is of so abstruse a character that to the ordinary mind the solution is enveloped in deepest mystery, but a method has been devised whereby anyone with a knowledge of the principles of elementary astronomy, geometry and plane trigonometry cannot fail to understand how the solution of this important problem is reached.

For an illustration of this method we have taken the transit of Mercury, which takes place on May 9th, 1891. The following data, which were taken from the American Ephemeris, are all that is required in the computation of its phases.

Declination of the Sun at Conjunction	+17° 32′ 1.3″
Declination of Mercury at Conjunction . .	+17° 18′ 1.6″
The right ascension of the Sun and Mercury at Conjunction	3^h 6^m 57.16^s
Hourly motion of the Sun in right ascension . .	+2′ 26.22″
Hourly motion of Mercury in right ascension . .	−1′ 18.49″
Hourly motion of the Sun in declination . . .	+0′ 39.54″
Hourly motion of Mercury in declination . . .	−1′ 6.67′
Sun's equatorial horizontal parallax	8.75′
Mercury's equatorial horizontal parallax . . .	15.84′
Sun's semidiameter	15′ 52.3″
Mercury's semidiameter	6.0″

The figure is drawn after the method of projecting lunar eclipses used in Loomis' Practical Astronomy. From C as a center with a radius equal to 15′ 52.3″, the semidiameter of the Sun, describe a circle ANBS which represents the disk of the Sun; through the center draw the line AB parallel to the equator, and the line NS perpendicular to AB; these are lines

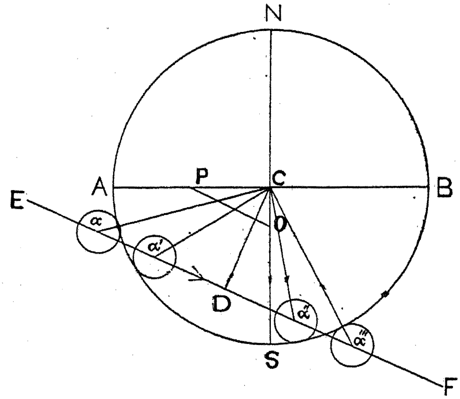

FIG. 40.

of right ascension and declination. The hourly motion of the Sun being eastward and that of Mercury being westward, the true hourly motion of Mercury relative to the Sun would be the sum of these motions, or 224.71″, and reducing this to motion in an arc of a great circle by multiplying by the cosine of the mean of the declinations of the Sun and Mercury we have 214.544″. Now from C lay off PC equal to 214.544″ or this motion in seconds of arc, and also OC on line NS equal to 106.21″, the motion in declination, and PO would be the resultant of the two motions in the direction of the planet's path across the disk of the Sun. Now at the time of conjunction the difference of declination of the two bodies would be this distance of the path from the center of the Sun, then lay off CG equal to 840.2″, then draw the line EF through G parallel to PO. This will be the path of Mercury as seen from the center of the earth with a telescope that does not invert. Then will a, a', a'', a''' be the positions of Mercury at first, second, third and fourth contacts. Draw lines $C\alpha$, $C\alpha'$, $C\alpha''$, $C\alpha'''$, and let fall a perpendicular from C to line EF. Then from the triangles thus formed, times of contact and the positions of the planet on the Sun with respect to the point N may be readily calculated by the methods of geometry and plane trigonometry.

In the right-angled triangle OCP we have given CP equal to 214.544″ and CO equal to 106.21″, and from the proportion,

$$CO : R :: CP : \tan COP,$$

we find angle COP equal to 63° 39′ 44.04″ and CPO equal to 26° 20′ 15.96″. Then by the "Law of Sines" we obtain OP equal to 239.394″. In the triangles OCP and CDG angle POC equals angle CGD, and angle OPC equals angle DCG, being angles of similar triangles. Geometry, theorem 12. From the proportion,

$$CG : CD :: R : \cos DCG$$

we find DC equal to 752.982″, and from the "Law of Sines" DG equals 372.764″. In triangle αDC we have given $C\alpha$ the semidiameter of the Sun plus that of Mercury equal to 15′ 58.3″ and DC equal to 752.982″. Then from the proportion

$$\alpha C : R :: DC : \sin D\alpha C$$

we find $D\alpha C$ equal to 51° 47′ 24.09″, and by a similar proportion we find $D\alpha$ equal to 592.751″, then the line αG equals αD plus DG, or 592.751″ plus 372.764″ equals 965.515″, and this divided by the hourly motion PO gives $4^h\ 1^m\ 59.365^s$, which is the time required for the planet to travel from α to G, and this subtracted from the time of conjunction, $15^h\ 57^m\ 22.224^s$, which is found by interpolation of differences, gives $11^h\ 55^m\ 22.859^s$ Greenwich time, (there is perhaps a slight error in this owing to the imperfection of the tables) which reduced to Central time would be $5^h\ 55^m\ 22.859^s$, the time of first contact as seen from the center of the earth. The time of second contact is computed in a similar manner from the triangle α', DC, $\alpha'C$, being 15′ 52.3″ minus 6.0″, and found to be $6^h\ 0^m\ 18.059''$. The times of third and fourth contacts are not calculated, not being visible at this place, but may be computed in the same manner. Now to an observer in north latitude the planet would appear to pass across the Sun's disk lower down than to an observer at the center of the earth owing to the parallax and from methods of computing parallax Loomis' Practical Ast. Arts, 210 and 211, we find the true time of first and second contacts for Columbia to be

$$5^h\ 54^m\ 0.0^s \text{ and } 5^h\ 58^m\ 55.29^s.$$

The next problem is to find the distances from the points of contact to N, which are the angles aCN and a'CN, which are the supplements of the angles aCG and a'CG. These supplements uncorrected for parallax equal

$$115^\circ\ 28'\ 40'' \text{ and } 116^\circ\ 24'\ 42''.$$

The first is useful in showing where to look for first contact.

SUPPLEMENTARY NOTE—The above results being computed for the latitude and longitude of Columbia, which is not very far from the geographical center of the State, are practically correct for all places in Missouri. The planet will not be visible to the naked eye, but may be seen with a small telescope, and should be looked for at ten minutes before six o'clock p. m., (railroad time) on the upper left hand limb of the Sun. The observed times of the contacts will agree with the computed times within one or two minutes, when the observations are accurate and the correct time used. I shall be glad to receive here at the observatory the results of any observations that may be made of this transit. Accurate daily time signals are received at the telegraph offices and railway depots throughout the State. MILTON UPDEGRAFF.

NOTE ON STADIA MEASURING.

BY PROF. W. H. ECHOLS, ROLLA, MO.

Let the stadia rod be, say, ten feet long and graduated in such a manner that one division corresponds to a distance of one foot from the external focus of the objective, as is usually the case. Let the graduations be numbered from the top down, and let a permanent target be set at that reading which represents the distance of the external focus of the objective from the center of the instrument, and in sighting on the telemeter always bring the top stadia wire on this target, then, if the line of sight be horizontal, the reading R of the bottom stadia wire is the distance in feet from the *center of the instrument.*

In order to obtain the horizontal distance from the rod to the instrument when the line of sight is inclined to the horizon at an angle θ, let the projecting sight rays of the upper and lower stadias make angles α_1 and α_2 with the line of sight of the instrument. Let the distances from the center of the instrument to the top and bottom stadia projections on the rod be ρ_1 and ρ_2 respectively. Then, if the rod be held vertical, and moved in the plane of the instrument so as to always give the same stadia reading R, we have from the figure,

$$\rho_1 = \frac{R}{\sin(\alpha_1 + \alpha_2)} \cos(\theta + \alpha_2); \quad \rho_2 = \frac{R}{\sin(\alpha_1 + \alpha_2)} \cos(\theta + \alpha_1),$$

so that these points of the rod describe vertical circles passing through the instrument I, and whose horizontal diameter is R (the difference from R being altogether too small to be considered under any circumstances, being only $\frac{1}{2}$R versine 2α; the cotan α being 200, and the maximum value of R, say 1000, makes this difference amount to only about one-fortieth of a foot).

In the figure then the reading on the vertical rod R for inclined lIne of sight θ, must be corrected by NM in order to give the true horizontal distance H of the rod from the instrument.

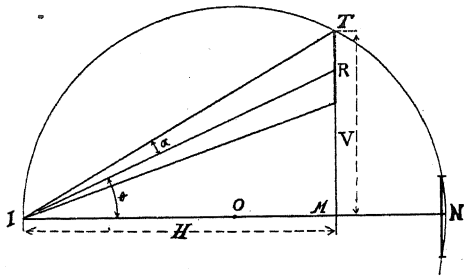

FIG. 41

$$NM = \tfrac{1}{2}R \text{ versine } 2\theta,$$
$$= R\sin^2\theta.$$

Which is the regular ordinary reduction formula. The value of this correction has been otherwise determined by the writer in the following simple manner.

Set a pointer sight on the rod so as to make with the upper part of the rod the angle $90^\circ + a_1$ (or a little more).

After the instrument man has read the vertical rod R, let the rodman swing the rod forward and give a reading, by means of his pointer, on the rod normal to the sight line of the upper stadia, call this reading r, then we have

$$\text{versine } \theta = \frac{r}{L},$$

where L is the length of the telememeter in graduations, say 1000, (owing to the fact that the top target is set a little below the zero of the rod, it is best to have a separate speaking scale for r, which is never more than a few tenths). Hence the correction is

$$dH = R\sin^2\theta = R\left\{2\frac{r}{L} - \left(\frac{r}{L}\right)^2\right\}.$$

In all ordinary work the second term in the brace may be neglected and the correction written

$$dH = \frac{2rR}{L} = \frac{rR}{500}.$$

Thus a rather useful reduction field formula is

$$H = R - \frac{rR}{500},$$

or if 100 graduations correspond to one foot of rod, then

$$H = R - \tfrac{1}{5} rR',$$

and the correction may be made mentally.

A good working rule is "the correction is *one foot* for each unit in r at 500′ away, and in proportion for other distances." The precision of the correction is the same as that of the distance observation, since an error of one rod division in reading r gives one foot error in the correction for mean distance of 500′, and the same error in reading R gives the same error of one foot in the distance H.

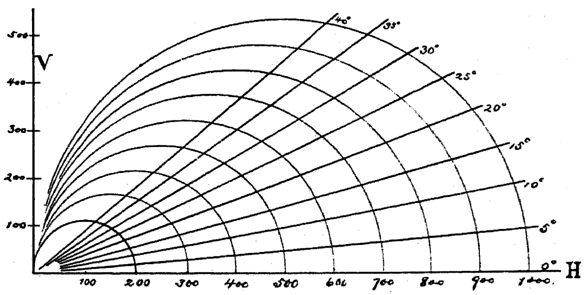

FIG. 42.

The above way of working up the stadia measures shows a rather interesting way of making a graphical reduction table which has the property of proportionality and at the same time permits the taking out of both the reduced horizontal and vertical distances at one reading of the pointer. The cut explains itself. The circles correspond to R readings and to distances, the rays to angular elevations; the V and H coordinates of any point determined by R and θ give the desired distances. Only angles up to 20° are really needed and the vertical scale may be magnified at will by orthogonally projecting the circles into ellipses. Such a table, however, is really not so useful as those based on the straight line graphical multiplication table, as shown in Mr. Baker's little work on surveying and elsewhere.

After correcting for the horizontal distance as above, one would naturally inquire for the corresponding correction for the vertical distance. The station is L feet below the top of the rod which is V feet above I and V is the geometric mean of the horizontal distance and its correction. We therefore become involved in the extraction of a root for finding V, which destroys the usefulness of any formula so derived.

Prof. Johnson in an interesting paper to *Engineering News* calls attention to the Porro telescope, in which, by the introduction of an additional lens, the stadia reading is made proportional to the distance from the center of the instrument instead of from the external focus of the objective. It is very doubtful that these telescopes will be constructed, for the introduction of an additional lens diminishes the definition by loss of light, etc., and the precision of the instrument is mainly dependent on this very property of the telescope, while the difficulty(?) of adding the instrumental constant may be otherwise obviated by targeting the rod as directed above.

This property of M. Porro's telescope was really secondary to that feature in it by which the rod-reading was rendered constant for all positions in the same vertical. Thus the instrument read the horizontal distance at once from a vertical rod. This was also effected by the introduction of an additional lens, which was so connected to the objective that the distance between their foci was made to vary (by a very simple arrangement) directly as the cosine of the inclination of the line of sight. See the Report of the U. S. Commissioners, Paris Universal Exposition, 1867. This construction M. Porro calls *stenallatic*, and when combined with the additional feature of referring distances to the center of the instrument he called the complete instrument the *anallatic* telescope.

A rather interesting formula for the vertical measure V may be gotten from the first figure, thus, we get easily

$$\tan \theta = \frac{M\cot \alpha_1 - N\cot\alpha_2}{R},$$

where M and N are the number of rod divisions between the horizontal wire and the upper and lower wire respectively ($M+N=R$). If, as is usually the case, $\cot\alpha_1=\cot\alpha_2=200$, then we have for the V measure,

$$V = H \tan\theta,$$

$$= \left\{ R - \frac{2rR}{L} \right\} 200 \frac{M-N}{R},$$

$$= 200(M-N) - 400(M-N)\frac{r}{L}.$$

For low values af θ the second term on the right is inappreciable, but unfortunately for application, M—N is at the same time so small that it cannot be measured with sufficient accuracy to give proper results.

For low values of θ, put circular measure for tan θ, then

$$V = \tfrac{7}{400} H \theta^{\circ}.$$

From these two values of V we find if $\theta = 10^{\circ}$ and $H = 480'$ then M—N is only $\frac{42}{100}$, about, of a rod division.

The last formula is simple enough for an ordinary field-working value of V, and when many reductions are to be made as in mapping, etc., the graphical tables are best.

ON "A NEW ELEMENTARY DEMONSTRATION OF THE PYTHAGOREAN PROPOSITION".

BY DR. ARTEMAS MARTIN, WASHINGTON, D. C.

The method of proof given on page 61, No. 2, of SCIENTIÆ BACCALAUREUS by Dr. Smith is *not* new. It has been published in many places and ascribed to various authors.

The demonstration in question was given in the *School Visitor*, Vol. 2, No. 4, April, 1881, p. 56, by William Hoover, then Superintendent of Schools, Wapakoneta, O., now Professor of Mathematics and Astronomy, Ohio University, as "*adapted from the French of Dalseme.*"

On page 159 of No. 5, Vol. 1, of the *Mathematical Monthly*, published February, 1859, this method of proof was given by Rev. A. D. Wheeler, of Brunswick, Me., without any reference.

In the same journal, Vol. 2, No. 2, October, 1859, pp. 45-52, Prof. John M. Richardson, Collegiate Institute, Boudon, Ga., gives a collection of twenty-eight demonstrations of this celebrated Theorem, among which, on p. 47, is the one under consideration, ascribed to Young. He mentions the collections of Camerer and Hoffmann; the former containing seventeen demonstrations and the latter, published in 1819, *thirty-three*.

Prof. Saradaranjan Ray of India gives the same demonstration on pp. 93-4 of Vol. 1 of his Geometry, and adds the following interesting historical note: "This proof is due to the Persian Astronomer Nasiruddin, who flourished in the 13th century under Jenghis Khan."

SOLUTIONS OF EXERCISES.

4.

The angles of depression of two towns, T and T′, n miles apart, are observed from a balloon and found to be *arc-cot* a and *arc-cot* a', respectively; the balloon moves in a line whose azimuth with respect to the line joining the two towns is *arc-cos* θ; upon arriving at a point known to be m miles (horizontally) from the first point of observation the angles of depression of T and T′ are now observed to be *arc-cot* b and *arc-cot* b' respectively. What was the height of the balloon at each station? [*Geo. R. Dean.*]

SOLUTION.

The figure is that of a quadrilateral whose sides are ha, ha', $h'b$, $h'b'$ (if h and h' be the respective heights of the balloon, whose diagonals are m and n making the angle arc-cos θ with each other.

Let the diagonals n and m divide each other into two segments x, y and z, u respectively. Then

(1). $x+y=n$; (2). $z+u=m.$

(3). $h^2a^2=x^2+z^2-2xz\theta,$

(4). $h^2a'^2=y^2+z^2+2yz\theta,$

(5). $h'^2b^2=x^2+u^2+2xu\theta,$

(6). $h'^2b'^2=y^2+u^2-2yu\theta.$

Divide (3) by (4) and (5) by (6) putting for brevity $a^2/a'^2=\rho$, $b^2/b'^2=\rho'$; then

(7). $\rho(y^2+z^2+2yz\theta)=x^2+z^2-2xz\theta,$

(8). $\rho'(y^2+u^2-2yu\theta)=x^2+u^2+2xu\theta.$

Put $n-y$ for x in (7) and (8), also put $m-z$ for u in (8), then these equations become

(9). $(\rho-1)(y^2+z^2+2yz\theta)=n^2-2ny-2nz\theta,$

(10). $(\rho'-1)(y^2+z^2+2yz\theta)=n^2+m^2+2mn\theta-\rho'm^2$
$+2(\rho'm\theta-m\theta-n)y$
$+2(\rho'm-n\theta-m)z.$

Eliminating $y^2+z^2+2yz\theta$ from these equations we get

(11). $Ay+Bz=C.$

where A, B and C are known.

In like manner put $n-x$ for y and $m-u$ for z in (7) and (8), whence result

(12). $(\rho-1)(x^2+u^2+2ux\theta)=m^2-\rho m^2-\rho n^2-2\rho mn\theta$
$+2(\rho m\theta+\rho n-m\theta)x$
$+2(\rho n\theta-\rho m-m)u.$

(13). $(\rho'-1)(x^2+u^2+2ux\theta)=-\rho'(n^2-2nx-2n\theta u)$

Eliminating $x^2+u^2+2ux\theta$ from these two equations we get

(14). $Dx+Eu=G.$

where D, E and G are known.

The four linear equations (1), (2), (11) and (14) solve the problem, since the values of x, y, u and z as determined from them in the usual manner, when substituted in (3) and (5) give h and h'.

In particular, take (4) from (3) and (6) from (5) and add the results, whence

$$h^2(a^2-a'^2)+h'^2(b^2-b'^2)=2mn\theta.$$

This gives the solution of the exercise as required in the text (Snowball's Trigonometry) where $h=h'$. [*W. H. Echols.*]

7.

On the sides of a triangle T, equilateral triangles are described, all outwards or all inwards. We thus get two new triangles T_1, T_2. Show that

(1) $\Delta_1+\Delta_2=5\Delta,$ where Δ, Δ_1, Δ_2 are the areas.

[*Frank Morley.*]

SOLUTION.

If the mid-points of the sides of T be joined we get another triangle T′ whose area is one-fourth that of T.

The straight line passing through a vertex of each triangle T_1, T_2 and T_1 is bisected by the vertex of T′. If the ends of this straight line describe the sides of T_1 and T_2 in the same time moving uniformly, its mid-point describes a corresponding side of T′. Hence we have by a well known theorem, for the relation between the areas of T_1, T_2 and T′,

$$\varDelta'=\tfrac{1}{4}\varDelta=\tfrac{1}{2}(\varDelta_1+\varDelta_2)-\tfrac{1}{4}\varOmega,$$
$$\therefore\quad \varDelta_1+\varDelta_2=\tfrac{1}{2}(\varDelta+\varOmega).$$

Where $\varOmega$ is the area of the figure described by a straight line, one end fixed, moving in a plane so as to always be equal in length and parallel in direction to the moving straight line above mentioned.

Let h_a, h_b, h_c be the altitudes of the equilateral triangles on the sides of T. Then the figure $\varOmega$ is easily seen to be composed of three triangles whose areas are

$$2h_ah_b \sin C, \quad 2h_bh_c \sin A, \quad 2h_ch_a \sin B.$$

Where A, B and C are the angles of T.

But $\quad h_a=a\sqrt{3}$, etc.

Hence $\quad \varOmega=6(ab \sin C+bc \sin A+ac \sin B)=9\varDelta$

Wherefore $\quad \varDelta_1+\varDelta_2=5\varDelta.$ [*Frank Bolles.*]

8.

In the Cassinian $r\,r_1=h^2$ the angle between the central radius and one focal radius is equal to that between the other focal radius and the normal. [*Frank Morley.*]

SOLUTION I.

Transforming to rectangular axes through the center we get for the equation to the oval

$$x^4+y^4+2x^2y^2-2a^2x^2+2a^2y^2=K^4-a^2.$$

(1). The equation to the normal is

$$y-y'=\frac{y}{x}\,\frac{x^2+y^2+a^2}{x^2+y^2-a^2}\,(x-x')$$

(2). Equation to one focal radius $y-y'=-\dfrac{y}{a-x}(x-x')$

(3). Equation to central radius $y-y'=\frac{y}{x}(x-x')$

(4). Equation to the other focal radius $y-y'=\frac{y}{a+x}(x-x')$

Angle between (1) and (2) is arc-tan $\dfrac{\frac{y}{x}\frac{x^2+y^2+a^2}{x^2+y^2-a^2}-\frac{y}{a-x}}{1+\frac{y}{x}\frac{x^2+y^2+a^2}{x^2+y^2-a^2}\frac{y}{a-x}}$

Angle between (3) and (4) is arc-tan $\dfrac{\frac{y}{x}-\frac{y}{a+x}}{1+\frac{y}{a+x}\frac{y}{x}}$

These angles are equal since the expressions for their tans are identical. [*Geo. R. Dean, T. U. Taylor.*]

SOLUTION II.

By the construction, due I believe to Tschirnhausen, for drawing the normal to any curve when its vectorial equation

$$f(r_1, r_2 \quad . \; . \; . \quad r_n)=0$$

is given, from the point on the curve lay off r_1 along r, and r along r_1, and join the ends of these lines; the normal bisects this joining line. Hence from a figure the truth of the proposition. [*Wm. P. Holman*]

10.

A 100 foot steel tape is longer than standard, so that at a certain temperature the tape measures a horizontal chord of 100 standard feet under a pull of 16 pounds supported at its ends. Find the pull that will give 40, 50 and in general D(< 100) standard foot horizontal chords, at same temperature, when the tape is supported at each end of the 40, 50, D foot graduations. [*W. O. Whitescarver.*]

SOLUTION.

Let L_0 be the true length of the (100′) tape of false length L feet, which subtends a chord of L true feet, under pull P.

Let L_0' be the true length of L' feet of the tape subtending the chord of L' standard feet, under pull P'.

Let $\mathbf{e}$ be the error of the tape, so that

$$L_0 = L + \mathbf{e}.$$

Then the error being uniformly distributed along the tape

$$L_0' = L' + \frac{L'}{L}\mathbf{e}.$$

Then
$$\frac{L_0}{L_0'} = \frac{L+\mathbf{e}}{L' + \frac{L'}{L}\mathbf{e}} = \frac{L\left(1+\frac{\mathbf{e}}{L}\right)}{L'\left(1+\frac{\mathbf{e}}{L}\right)} = \frac{L}{L'}.$$

The relation between the true length of the tape, the chord, the pull and its weight W, is

(1).
$$\frac{L_0}{L} = 1+\tfrac{1}{24}\left\{\frac{W}{P}\right\}^2.$$

(2). Also
$$\frac{L_0'}{L'} = 1+\tfrac{1}{24}\left\{\frac{W'}{P'}\right\}^2 = 1+\tfrac{1}{24}\left\{\frac{L'}{L}\,\frac{W}{P'}\right\}^2$$

Where $W' = \frac{L'}{L}W$ is the weight of the portion of tape used.

Dividing (1) by (2)

$$\frac{L_0}{L'_0}\,\frac{L'}{L} = \frac{24+\left\{\frac{W}{P}\right\}^2}{24+\left\{\frac{L'}{L}\,\frac{W}{P'}\right\}^2}$$

or
$$\frac{L}{L}\,\frac{W}{P'} = \frac{W}{P}$$

$$P' = \frac{L'}{L}P.$$

[*A. J. Stewart, George Herdman and others.*]

11.

A particle is set free at the highest point of a smooth sphere which stands on a horizontal plane. The particle slightly disturbed begins to move in a certain direction. Where does it meet the plane and what is the duration of motion?

[*Elmo G. Harris.*]

SOLUTION.

The particle will describe the arc of a vertical great circle if acted on by no other force than gravity. Its velocity at any point on the sphere is $\sqrt{2gh}$ where h is the vertical descent.

Let θ be the angle which the radius to the particle makes with the vertical through the sphere center, then the duration of motion on the sphere is (since the particle leaves the sphere at a point where θ=arc-cos $\frac{2}{3}$), a being sphere radius.

$$t=\int_0^{\frac{1}{3}a}\frac{ds}{\sqrt{2gh}}=\tfrac{1}{2}\sqrt{\frac{a}{g}}\int_0^{\text{arc-cos}\,\frac{2}{3}}\frac{d\theta}{\sin\frac{1}{2}\theta}$$

$$=\tfrac{1}{2}\sqrt{\frac{a}{g}}\log\tan\tfrac{1}{4}\,\text{arc-cos}\,\tfrac{2}{3}.$$

At this point as origin refer the particle after leaving the sphere to vertical and horizontal axes of coordinates whose plane through the center of the sphere is that of the trajectory. The coordinates of the particle's position at t' seconds after leaving the sphere is determined by

$$x=Vt'\sin\theta,$$

$$=\tfrac{2\sqrt{5}}{9}agt'.$$

$$y=Vt'\cos\theta+\tfrac{1}{2}gt'^2$$

$$=\tfrac{4}{9}agt'+\tfrac{1}{2}gt'^2.$$

Where t' determined from

$$t'^2+\tfrac{8}{9}at'=\tfrac{10}{3}\frac{a}{g},$$

gives the duration of motion to the horizontal plane, and when substituted in x and y above determines the point where it strikes the plane.

[*T. U. Taylor.*]

12.

A smooth tube bent to the shape of a semi-ellipse is fixed in a vertical plane, its major axis horizontal, its semi-minor axis upward. A heavy flexible string passing through the tube and hanging at rest is cut at one end of the tube. What is the velocity of the string as it leaves the tube? [*W. H. Echols.*]

SOLUTION.

Let μ equal weight of unit length of string; P be any position of the end of string which starts from A'.

Let s equal arc A'P; l length of string hanging vertically when P is at A'; l' variable length $l+s$ at any position of P; F accelerating force at any position of P.

$$\text{Then } F=\mu l'+\mu\int_0^s ds\sin\theta=\mu l'+\mu\int_0^y ds\frac{dy}{ds},$$

$$=\mu(l'+y)=\mu(l+s+y).$$

$$\tfrac{1}{2}m\,v^2=\int_0^s F ds,\qquad v^2=\frac{2\mu}{m}\int^S(l+s+y)ds.$$

(S equals length of tube.) $$=\frac{2\mu}{m}\left\{lS+\tfrac{1}{2}S^2+\int_0^S yds\right\}.$$

In the equation to the ellipse let y equal $b\cos\varphi$.

Then $$ds=a\sqrt{1-\varepsilon^2\sin^2\varphi}\,d\varphi.$$

$$\int_0^S yds=2ab\int_0^{\frac{\pi}{2}}\left\{1-\varepsilon^2\sin^2\varphi\right\}^{\frac{1}{2}}\cos\varphi d\varphi,$$

$$=2ab\varepsilon\int_0^{\frac{\pi}{2}}\left\{\frac{1}{\varepsilon^2}-\sin^2\varphi\right\}^{\frac{1}{2}}\cos\varphi d\varphi,$$

$$=ab\varepsilon\left\{\sin\varphi\sqrt{\frac{1}{\varepsilon^2}-\sin^2\varphi}+\frac{1}{\varepsilon^2}\sin^{-1}(\varepsilon\sin\omega)\right\}_0^{\frac{1}{2}\pi},$$

$$=ab\varepsilon\left\{\sqrt{\frac{1}{\varepsilon^2}-1}+\frac{1}{\varepsilon^2}\sin^{-1}\varepsilon\right\},$$

$$=ab\left\{\sqrt{1-\varepsilon^2}+\frac{1}{\varepsilon}\sin^{-1}\varepsilon\right\}=b^2+\frac{ab}{\varepsilon}\sin^{-1}\varepsilon.$$

$$\therefore v^2=\frac{2\mu}{m}\left\{lS+\tfrac{1}{2}S^2+b^2+\frac{ab}{\varepsilon}\sin^{-1}\varepsilon\right\},\quad \text{Since } m=\mu\frac{l+s}{g}$$

$$v=\sqrt{\frac{2g}{l+S}}\left\{lS+\tfrac{1}{2}S^2+b^2+\frac{ab}{\varepsilon}\sin\varepsilon^{-1}\right\}^{\frac{1}{2}}.$$

[*Wm. P. Holman.*]

14.

Given on the ground a circular curve of known radius intersecting a given straight line at a given point and given angle; it is required to unite the two by another circular curve of given radius in such a manner as to have a common tangent of length t between the curves. [*W. H. Echols.*]

SOLUTION.

Let D be the distance between the centers.

t be the length of given tangent.

α the angle of intersection.

R, r radii of the curves.

T distance of vertex of a from the P.T.

Then $$D^2=(R+r)^2+t^2=(T+r\sin\alpha)^2+(R-r\cos\alpha)^2.$$

Whence $$T=r\sin\alpha\left\{\pm\sqrt{1+\frac{t^2+2Rr(1+\cos\alpha)}{r^2\sin^2\alpha}}-1\right\}.$$

The central angle of the R curve is $\varphi-\Psi$ where

$$\tan\Psi=\frac{t}{R+r};\quad \sin\varphi=\frac{T+R\sin\alpha}{D}$$

If $t=0$, we have solution of Exercise 13.

$$T=r\sin\alpha\left\{\pm\sqrt{1+\frac{2R(1+\cos)}{r^2\sin\alpha}}-1\right\},$$

and for the central angle θ,

$$\sin\theta=\frac{T+R\sin\alpha}{R+r}.$$

[*Geo R. Dean. Also solved by T. U. Taylor.*]

15.

$$\int(a^2-x^2)\text{arc-cos}\left(\frac{a}{2\sqrt{a^2-x^2}}\right)dx.$$ [*W. H. Echols.*]

SOLUTION.

Integrating by parts

$$\int(a^2-x^2)\text{arc-cos}\left[\frac{a}{2\sqrt{a^2-x^2}}\right]dx$$

$$=\int \text{arc-cos}\left[\frac{a}{2\sqrt{a^2-x^2}}\right]d(a^2-\frac{x^2}{3})x$$

$$=x(a^2-\frac{x^2}{3})\ \text{arc-cos}\left[\frac{a}{2\sqrt{a^2-x^2}}\right]-a^3\int\frac{x^2dx}{\sqrt{(a^2-x^2)(3a^2-4x^2)}}$$

$$-\frac{a}{3}\int\frac{x^4dx}{\sqrt{(a^2-x^2)(3a^2-4x^2)}}$$

The two last integrals are known forms, and give

$$\int\frac{x^2dx}{\sqrt{(a^2-x^2)(3a^2-4x^2)}}=-\tfrac{1}{2}\text{arc-sin}\frac{2x}{a\sqrt{3}}+\text{arc-tan}\frac{x}{\sqrt{3a^2-4x^2}}$$

$$\int\frac{x^4dx}{\sqrt{(a^2-x^2)(3a^2-4x^2)}}=\frac{x\sqrt{3a^2-4x^2}}{8}-\frac{11a^2}{16}\text{arc-sin}\ \frac{2x}{a\sqrt{3}}$$

$$+a^2\text{arc-tan}\ \frac{x}{\sqrt{3a^2-4x^2}}.$$

$$\therefore\int(a^2-x^2)\text{arc-cos}\left[\frac{a}{2\sqrt{a^2-x^2}}\right]dx$$

$$=\frac{ax\sqrt{3a^2-4x^2}}{24}+\frac{13a^3}{48}\text{arc-sin}\ \frac{2x}{a\sqrt{3}}$$

$$+x(a^2-\frac{x^2}{3})\ \text{arc-cos}\left[\frac{a}{2\sqrt{a^2-x^2}}\right]-\frac{2a^3}{3}\text{arc-tan}\ \frac{x}{\sqrt{3a^2-4x^2}}.$$

[*Wm. E. Heal.*]

EXERCISES.

21.

A horizontal beam, span a, resting on two supports at ends is loaded so that the load per running foot varies as the square of the distance from one support. Find the tangent to the elastica at each end of the beam and the maximum deflection and that at the center. [*T. U. Taylor.*]

22.

A horizontal beam, span a, resting on two supports at its ends has the shape of a right circular cone whose axis is horizontal. Find the central deflection. [*T. U. Taylor.*]

23.

$$\int\int \frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}.$$ [*G. H. Harvill.*]

24.

In any triangle the rhombi on a and b with angle C are together equal to the rhombus on c with altitude in the same ratio to c as that of the diagonals of the rhombi on a and b.

[*W. H Echols.*]

25.

Let O be the orthocenter of the triangle ABC and D, E and F feet of the perpendiculars from A, B and C on the opposite sides respectively. Show that the areas of the triangles BOD, COE, AOF are together equal to the areas of DOC, OEA and FOB taken together. [*W. H. Echols.*]

26.

An elastic ring is gently placed on a smooth vertical cone of revolution. Find the position of equilibrium and the lowest position of descent. Also determine the time of vibration.

[*W. H. Echols.*]

27.

In any quadrilateral the sum of the squares on the lines joining the mid-points of the sides is equal to the squares on the diagonals. [*Sallie Millard.*]

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