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THE SCIENCE ABSOLUTE OF SPACE,

INDEPENDENT OF THE TRUTH OR FALSITY OF EUCLID'S AXIOM XI (WHICH NEVER CAN BE ESTABLISHED A PRIORI);

Followed by the geometric quadrature of the circle in the case of the falsity of Axiom XI,

BY

John Bolyai,

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TRANSLATED INTO ENGLISH

BY

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TRANSLATOR'S INTRODUCTION.

Through all its editions up to the last, America's favorite geometry, Wentworth's, taught in all seriousness the following proposition (see 3d edition, 1887, §387, page 224): *To inscribe a regular polygon of any number of sides in a given circle.* But in this, as in some other respects, the book was only more than two thousand years behind the times. Euclid would have smiled at the unconsciousness with which this American Jonah swallowed his impossible whale. Euclid could inscribe regular polygons of 3, 4, 5, 15 sides or numbers obtained by doubling these. Those of 7, 9, 11, 13, 14 sides no man ever could or ever will geometrically inscribe. When on the evening of March 30th, 1796, Gauss showed to his student friend, the Hungarian, Wolfgang Bolyai, the formula which gave the geometric inscription of the regular polygon of 17 sides, it was with the remark that this alone could be his epitaph, if it were not a pity to omit so much that went with it.

Was it this break beyond Euclid's enchanted bounds that started these two young men in that re-sifting of the very foundations of geometry which led to those new conceptions of the whole subject just now, after another hundred years, beginning to be taught in America's foremost universities?

Wolfgang Bolyai was born February 9th, 1775, in that part of Transylvania called Székelyföld. He studied first at Enyed, afterward at Klausenburg, and in 1796, with a son of Baron

Simon Kemény, went first to Jena, then to Göttingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life. The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss.

Gauss said that Bolyai was the only man who completely understood his views on the metaphysics of mathematics. Everyone who met him felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written to Gauss in 1801 that Bolyai was one of the most extraordinary men he had ever known.

On his return home in 1802 Bolyai was made professor of mathematics in the Reformed College of Maros-Vásárhely.

Here during the 47 years of his active teaching he had for scholars most of the present professors in Transylvania, and nearly all the nobility of the country.

Sylvester has said that mathematics is nearest akin to poetry. Bolyai's first works published were dramas, and translations of English and German poetry into Hungarian.

In 1830 he published an arithmetic. Then came his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, with title as follows:

Tentamen juventutem studiosam in elementa matheseos purae, elementaris ac sublimioris, methodo intuitivo, evidentiaque huic propria, introducendi. Cum Appendice triplici.

Auctore Professore Matheseos et Physices Chemiaeque publico ordinario.

Tomus primus. Maros Vásárhelyini, 1832.

Tomus secundus. 1833.

The first volume contains:

Preface of two pages: *Lectori salutem.*

A folio table: *Explicatio signorum.*

Index rerum (I—XXXII). *Errata* (XXXIII—LXXIV). *Errores recentius detecti* (LXXIV—XCVIII).

Now comes the body of text (pages 1 502). Then with special paging and a new title page, comes the immortal appendix compos d by John Bolyai, son of Wolfgang:

APPENDIX *scientiam spatii absolute veram exhibens: a veritate aut falsitate axiomatis* XI *Euclidei* (*a priori haud unquam decidenda*) *independentem; adjecta, ad casum falsitatis, quadratura circuli geometrica. Auctore* JOANNE BOLYAI *de eadem, Geometrarum in Exercitu Caesaris Regio Austriaco Castrensium Capitaneo.* Twenty-six pages of text, two pages of errata.

Finally (pages I XVI), in Hungarian, the names of the subscribers, the nomenclature, and additions to this volume by W. Bolyai. Then 4 plates of figures, the first 3 pertaining to the body of the text, the last to the Appendix.

It is this Appendix which we now give for the first time in English. Milton received but a paltry 5 pounds for his Paradise Lost; but it was at least plus 5, John Bolyai, as we learn from volume second, page 384, of the *Tentamen*, contributed, for the printing of his eternal 26 pages, 104 florins 54 kreuzers.

His father, treating in the body of the work the theory of parallels, says, *à propos* of the systems which are possible when we contradict Euclid's axiom XI, "Appendicis Auctor, rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit, quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis."

And again: "Nihilominus tamen quaestio suboritur: quid si novum axioma detur, per quod determinetur *u?* Tentamina idcirco, quae olim feceram, breviter exponenda veniunt, ne saltem alius quis operam eodem perdat." He speaks of his son's beautiful treatise with natural admiration: Thus, Vol. I,

p. 502, Nec operae pretium est plura referre; quum res tota ex altiori contemplationis puncto, in ima penetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno digna legi.

And Vol. II, page 380, "Denique aliquid Auctori Appendicis . . . addere fas sit: quo tamen ignoscat, si quid non acu ejus tetigerim."

This wonderful production of pure genius, this Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriptive adjective, *Euclidean*, this strange Hungarian flower was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. In the second edition of his Elemente der Mathematik in 1867, Dr. Baltzer called attention to this re-making of Geometry, and the name Bolyai was at last given its place in the history of science. Before that, the father Wolfgang Bolyai seems to have been the only person who really appreciated the work of the son John Bolyai. He refers to it in a subsequent work printed in 1846, Uertan elemei kezdöknek, figures for which, we learn, were drawn by his grandson, John's son. Then comes his last work, the only one composed in German, entitled:

Kurzer Grundriss eines Versuchs:

I. *Die Arithmetik*, durch zweckmæssig construirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der *Geometrie*, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krummen, der verschiedenen Arten der Gleichheit *u. d. gl.* nicht nur scharf zu bestimmen, sondern auch ihr Seyn im Raume zu beweisen; und da die Frage, *ob zwey von der dritten geschnittenen Geraden, wenn die Summe der inneren*

Winkel nicht=2*R*, *sich schneiden oder nicht?* niemand auf der Erde ohne ein Axiom (wie Euclid das XI) aufzustellen, beantworten wird; die davon unabhængige Geometrie abzusondern, und eine auf die Ja–Antwort, andere auf das Nein so zu bauen, dass die Formeln der letzten, auf einen Wink auch in der ersten gültig seyen.

Nach einem lateinischen Werke von 1829, M. Vàsàrhely; und eben da selbst gedruckten ungarischen:

Maros-Vàsàrhely, 1851, 88 pages of text.

In this he says, referring to his son's Appendix scientiam spatii ubsolute veram exhibens; "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . . From Goettingen the giant of mathematics, who from his pinnacle embraces in the same view the stars and the abysses, wrote that he was charmed to see executed the work which he had commenced, only to leave it after him in his papers."

On the 9th of March, 1832, Wolfgang Bolyai was made corresponding member in the mathematics section of the Hungarian Academy. As professor he exercised a powerful influence in his country. In his private life he was a type of true originality. He wore roomy black Hungarian pants, a white flannel jacket, high boots, and a broad hat like an old-time planter's. The smoke-stained wall of his antique domicile was adorned by pictures of his friend Gauss, of Schiller, and of Shakespeare, whom he loved to call the child of Nature. His violin was a constant solace. He died the 20th of November, 1856. He ordered that his grave should bear no mark.

His son John died in 1860, seven years before the world began to know of his unique and wonderful work. He was born at Klausenburg, in Transylvania, the 15th of December, 1802.

He studied in one of the institutions founded in Transylvania by the Imperial Academy of Engineering of Vienna, and graduated the 7th of September, 1822, as cadet of engineers. The first of September, 1823, he was made second lieutenant, and the 16th of June, 1833, he was put on the retired list as captain. His profound mathematical ability showed itself physically not only in his handling of the violin, where he was a master, but also of arms, where he was unapproachable. It was this skill which caused his being retired so early from the army, though it saved him from the fate of a kindred spirit, the lamented Galois, killed in a duel when only 19. Bolyai when in garrison with cavalry officers was challenged by 13 of them at once. He accepted all, only stipulating that between each duel he might play a bit on his violin. He was victor thirteen times.

Beyond the *Appendix*, whose translation into English is here given, John Bolyai published nothing; and the thousand pages of manuscript which he left have never been read by a competent mathematician. They are in the library of the Reformed College of Maros-Vàsàrhely. We hear that he had conceived the project of working out a universal language, akin to that which music has, or that of mathematics.

If in this he was only an anticipator of Volapük, we think nothing of it; but it rather seems that he was another Boole, and if so, what discoveries in algorithmic logic might lie hidden in his papers!

In 1853 he must have thought of printing part of his mathematical works, for he left parts of a book with the title:

Principia doctrinae novae quantitatum imaginariarum perfectae uniceque satisfacientis, aliaeque disquisitiones analyticae et analytico-geometricae cardinales gravissimaeque; auctore Johan. Bolyai de eadem, C. R. austriaco castrensium captaneo pensionato.

Vindobonae, vel Maros-Vàsàrhelyini, 1853.

To him who hath shall be given, and it would be natural enough if the world still gives to Gauss, the greatest and best known mathematician of his generation, some of the credit which really belongs to the name of Bolyai. On the completion of his mathematical studies at the university, the Georgia Augusta, Bolyai left Goettingen the 5th of June, 1799.

From Braunschweig, Gauss writes to him in Klausenburg at the end of the year:

"I very much regret that I did not make use of our former proximity to find out *more* of your investigations in regard to the first grounds of geometry; I should certainly thereby have spared myself much vain labor, and would have become more restful than any one such as I can be, so long as, on such a subject, there yet remains so much to be wished for. In my own work thereon I myself have advanced far (though my other wholly heterogeneous employments leave me little time therefor), but *the* way, which I have hit upon, leads not so much to the goal which one wishes, as much more to making doubtful the truth of geometry. I have hit upon much which, with most, would pass for a proof, but which in my eyes proves as good as nothing. For example, if one could prove that a rectilineal triangle is possible whose content may be greater than any given surface, then am I in condition to prove with perfect rigor all geometry. Most would indeed let that pass as an axiom; I not; it might well be possible, that, how far apart soever one took the three vertices of the triangle in space, yet the content was always under a given limit. I have more such theorems, but in none do I find anything satisfying."

From this letter we see that in 1799 Gauss was still trying to prove *a priori* the eternal reality of the Euclidean system, what John Bolyai calls the system Σ. Some time in the next thirty years he comes to Bolyai's conclusion, for in 1829 he writes to Bessel as follows:

"At times in certain free hours, I have meditated again on a theme which, with me, is already nearly 40 years old, I mean the first grounds of geometry. I do not know whether I have spoken to you of my views thereupon. Here also have I much still further consolidated, and my conviction that we cannot found geometry completely *a priori*, has become, if possible, still firmer. Meanwhile, I am still far from attaining to the working out of my *very extended* researches for publication, and perhaps that will never happen in my lifetime, for I dread the outcry of the opposition if I should express my views *fully*."

Later Gauss adds:

"According to my deepest conviction, the science of space has to our science of necessary truths a relation wholly different from the pure science of quantity; there is lacking to our knowledge of the former (space lore) throughout, *that* complete persuasion of its necessity (consequently also of its absolute truth) which is peculiar to the *latter;* we must in humility admit, that, if number is *merely* a product of our mind, space has also a *reality beyond* our mind, of which we cannot fully foreordain the laws *a priori*."

More than twenty years after this, Gauss heard from his own pupil, Riemann, the marvelous dissertation which to Bolyai's spaces, got by denying the axiom of parallels, added as many others got by denying the infinite size of the straight line.

Beltrami showed ("Saggio di interpretazione della geometria non-euclidea," Giorm di Matematiche, 1868) that Bolyai's geometry in a plane is equivalent to the Euclidean geometry on a surface of constant negative curvature. Riemann's finite space, of positive curvature, was studied by Felix Klein (1871–2, Math. Annalen IV & VI), and by him named *Elliptic*, while Euclid's he called *Parabolic*, and Bolyai's *Hyperbolic*. I notice that our new Century dictionary confuses Hyperbolic with Elliptic geometry, giving to each the definition of the other.

Cayley carried on the subject to trigonometry in an article entitled, "On the Non-Euclidean Geometry (Mathematische Annalen, v. pp. 630–4, 1872(, which begins as follows: "The theory of the Non-Euclidean Geometry as developed in Dr. Klein's paper "Ueber die Nicht-Euclidische Geometrie" may be illustrated by showing how, in such a system, we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the "hyperbolic" case of plane geometry; viz. the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to points *within* the circle.

I use the simple letters a, A, . . to denote (linear or angular) distances measured in the ordinary manner; and the same letters with a superscript stroke, $\bar{a}$, $\bar{A}$, . . to denote the same distances measured according to the theory." His result is "that the formulæ are in fact similar to those of spherical trigonometry with only cos h $\bar{a}$, sin h $\bar{a}$, etc., instead of cos a, sin a, etc."

In my first paper on the Bibliography of Hyper-Space and Non-Euclidean Geometry (American Journal of Mathematics, Vol. I, No. 3, pp. 261–276, 1878), I mentioned also Réthy's article: Die Fundamental Gleichungen der nicht-euklidischen Trigonometrie auf elementarem Wege abgeleitet:

Grunert's Archiv, LVIII, 416; also a number of works carrying these ideas on into mechanics.

EXPLANATION OF SIGNS.

The straight ABC means the aggregate of all points situated in the same straight line with A and B.

The sect AB means that piece of the straight AB between the points A and B.

The ray AB means that half of the straight AB which commences at the point A and contains the point B.

The plane ABC means the aggregate of all points situated in the same plane as the three points (not in a straight) A, B, C.

The hemi-plane ABC means that half of the plane ABC which starts from the straight AB and contains the point C.

$\angle ABC$ means the smaller of the pieces into which the plane ABC is parted by the rays BA, BC, or the non-reflex angle of which the sides are the rays BA, BC.

ABCD (the point D being situated within $\angle$ABC, and the straights BA, CD not intersecting) means the portion of $\angle$ABC comprised between ray BA, sect BC, ray CD, while BACD designates the portion of the plane ABC comprised between the straights AB and CD.

$\perp$ is the sign of perpendicularity.

$\parallel$ is the sign of parallelism.

$\angle$ means angle.

rt. $\angle$ is right angle.

st. $\angle$ is straight angle.

$\cong$ is the sign of congruence, indicating that two magnitudes are superposable.

AB $\triangleq$ CD means $\angle$CAB$=\angle$ACD.

$x \doteq a$ means x converges toward the limit a.

$\triangle$ is triangle.

$\bigcirc r$ means the [circumference of the] circle of radius r.

$\odot r$ means the area of the surface of the circle of radius r.

The Science of Absolute Space.

1. If the ray AM is not cut by the ray BN, situated in the same plane, but is cut by every other ray BP comprised in the angle ABN, we will call ray BN *parallel* to ray AM, that is to say we will have BN || AM.

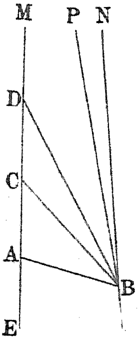


FIG. I.

It is easy to see that *there is one such ray BN, and only one*, passing through any point B (taken outside of the straight AM), and that the sum of the angles BAM, ABN cannot exceed a st. ∠. Because, in moving BC around B until BAM+ABC=st. ∠, there will be an instant where ray BC will *commence* not to cut ray AM, and it is then that we have BC || AM. It is clear, at the same time, that BN || EM, whatever be the point E taken on the straight AM.

If while the point C goes away to infinity on ray AM, we take always CD=CB, we will have constantly CBD=CDB <NBC. Now NBC≐O; therefore also ADB≐O.

2. If BN || AM, we will have also CN || AM. Take D any point of MACN. If C is on ray BN, ray BD will cut ray AM, since BN || AM. Therefore ray CD will also cut ray AM. If C is situated on ray BP, take BQ || CD; BQ will fall within the ∠ABN (§1), and consequently will cut ray AM; therefore ray CD will also cut ray AM. Therefore every ray CD (in ACN) cuts, in each case, the ray AM, without CN itself cutting ray AM. Therefore we have always CN || AM.

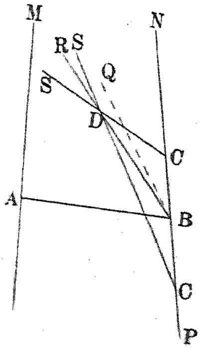


FIG. 2.

3. If BR and CS are each || AM, and C is not situated on the straight BR, then ray BR and ray CS do not intersect. Because if ray BR and ray CS had a common point D, then (§ 2) DR and DS would be each || AM, ray DR (§ 1) would coincide with ray DS, and C would fall on the straight BR, which is contrary to the hypothesis.

4. If MAN>MAB, we will have, for every point B of ray AB, a point C of ray AM, such that BCM=NAM.

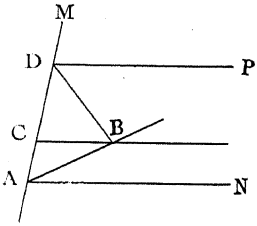


FIG. 3.

For, (§ 1), we may draw BD so that BDM >NAM, and making MDP=MAN, B will be contained in NADP. If therefore we carry NAM along AM, until ray AN arrives on ray DP, ray AN will have necessarily passed through B, and somewhere we have had BCM=NAM.

5. If BN || AM, there is on the straight AM a point F such that FM $\triangleq$ BN. For we can get (§ 1) BCM> CBN, and if CE=CB, it follows that EC $\triangleq$ BC, whence BEM< EBN. Move the point P on EC. The angle BPM, for P near E, will commence by being < the corresponding angle PBN, and for P near C, it will finish by being >PBN. Now the angle BPM increases continuously from BEM to BCM, since (§ 4) there exists no angle >BEM and < BCM, to which BPM can not become equal. Likewise PBN decreases continuously from EBN to CBN. There is therefore on EC a point F such that BFM=FBM.

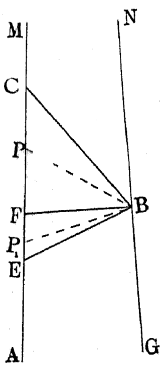


FIG. 4.

6. If BN || AM and E any point of the straight AM, and G any point of the straight BN, then GN || EM and EM || GN.

Because we have (§ 1) BN || EM, whence (§ 2) GN || EM. If now we make (§ 5) ∠BFM=∠FBN, then MFBN$\cong$NBFM, and consequently, since BN || FM, we have also FM || BN, and from what precedes EM || GN.

7. If BN and CP are each || AM, and C not on the straight BN, we shall have also BN || CP.

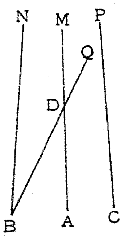


FIG. 5.

The rays BN and CP do not intersect (§ 3). Moreover, AM, BN and CP either are or are not in the same plane, and in the first case, AM either is or is not within BNCP.

1. If AM, BN, CP are in the same plane, and AM falls within BNCP, then every ray BQ drawn within ∠NBC will cut the ray AM somewhere in D, since BN || AM. Moreover, since DM || CP (§ 6), the ray DQ will cut the ray CP, therefore BN || CP.

2. If BN and CP are on the same side of AM, one of them, for example CP, will be contained between the two other straights BN, AM.

Now, every ray BQ within ∠NBA meets the ray AM; consequently it also meets CP. Therefore BN || CP.

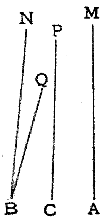


FIG. 6.

3. If the planes MAB, MAC make an angle, then CBN and ABN can have in common nothing but the straight BN, while the straight AM (in ABM) will have nothing in common with the ray BN, and in consequence, also NBC will have nothing in common with the straight AM.

Now every hemi plane BCD, drawn through the ray BD (situated in ∠NBA), will meet the ray AM, since ray BQ meets ray AM (as BN || AM). Therefore in revolving the hemi-plane BCD around BC until this hemi-plane *begins* to leave the ray AM, the hemi-plane BCD will come into coincidence with the hemi-plane BCN. By parity of reasoning this same hemi-plane will come into coincidence with hemi-plane BCP. Therefore BN is in the plane

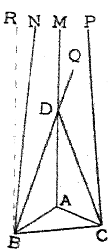


FIG. 7.

BCP. Moreover, if BR ∥ CP, then (AM being also ∥ CP) BR will be by the same reasoning, in the plane BAM, and also (since BR ∥ CP) in the plane BCP. Therefore the straight BR, being common to the two planes MAB, PCB, is identical with the straight BN. Therefore BN ∥ CP.*

If therefore CP ∥ AM, and B exterior to the plane CAM, then the intersection BM of the planes BAM, CAP is ∥ at the same time to AM and CP.

8. If BN ∥ CP and ∠CBN=∠BCP, and AM (in NBCP) is ⊥ the sect BC at its mid point, then BN ∥ AM.

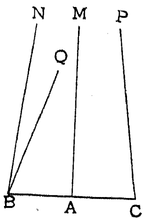


FIG. 8.

For, if ray BN met ray AM, then ray CP would also meet ray AM at the same point (because MABN≌MACP), and this would be common to the rays BN, CP themselves, while on the contrary BN ∥ CP. Moreover every ray BQ interior to ∠CBN meets ray CP; therefore also it meets ray AM. Consequently BN ∥ AM.

9. If BN ∥ AM, and MAP ⊥ MAB, and the dihedral ∠DNBA of the planes NBD, NBA (prolonged on that side of MABN where MAP is) is <rt. ∠, then MAP and NBD intersect.

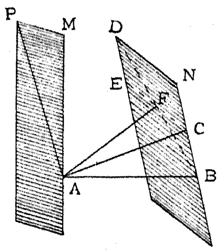


FIG. 9.

Make ∠BAM=rt. ∠, and AC ⊥ BN (whether or not C coincides with B), and CE ⊥ BN (in NBD).

We shall have by hypothesis ∠ACE <rt. ∠, and AF (⊥ CE) will fall within ∠ACE.

Let ray AP be the intersection of the hemi-planes ABF, AMP (which have the point A common). We shall have (since BAM ⊥ MAP) ∠BAP=∠BAM=rt. ∠.

*In placing this third case before the other two, these could be demonstrated with more brevity and elegance, like case 2 of §10. (Author's note.)

If now we move the hemi-plane ABF around the fixed points A and B until it coincides with the hemi-plane ABM, then ray AP will fall on ray AM, and since AC ⊥ BN, and sect AF< sect AC, then sect AF will have its extremity between ray BN and ray AM, and consequently BF will fall within ∠ABN. Now, *in this position*, ray BF will meet ray AP (since BN ∥ AM); therefore ray AP and ray BF intersect also *in the original position*, and the point of meeting is common to the hemi-planes MAP and NBD. Therefore the hemi-planes MAP and NBD intersect. From this we may deduce that the hemi-planes MAP and NBD intersect whenever the sum of the dihedral angles which they make with MAB is <rt. ∠.

10. If BN ∥ AM, and CP ∥ AM, and ∠ABN=∠BAM and ∠ACP=∠CAM, then also BN ∥ CP and ∠BCN=∠BCP.

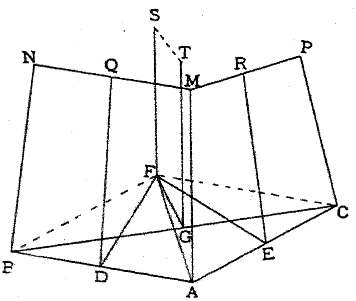


FIG. 10.

For, either the planes MAB, MAC make an angle, or they form one and the same plane.

I. In the first case, draw the hemi-plane QDF ⊥ sect AB at its mid point. Then we will have DQ ⊥ AB and consequently DQ ∥ AM (§ 8). Likewise if hemi-plane ERS is ⊥ sect AC at its mid point, ER ∥ AM.

Consequently (§ 7) DQ ∥ ER.

Hence (§ 9) the hemi-planes QDF and ERS intersect, and have (§ 7) their intersection the ray FS ∥ DQ. Moreover since BN ∥ DQ, we have also FS ∥ BN. Besides for every point F of FS we have the sects FB=FA=FC, and so the ray FS is in the hemi-plane TGF ⊥ sect BC at its mid point. Now since FS ∥ BN we have (§ 7) GT ∥ BN. In the same way GT ∥ CP. But GT ⊥ sect BC at its mid point.

But GT $\perp$ sect BC at its mid point. Therefore TGBN $\cong$ TGCP (§ 1), and BN $\parallel$ CP and $\angle$CBN=$\angle$BCP.

2. If BM, AM, and CP are in one and the same plane, let FS be exterior to this plane and FS $\parallel$ AM, and $\angle$AFS=$\angle$FAM. Then from what precedes, FS $\parallel$ BN, FS $\parallel$ CP, $\angle$BFS=$\angle$FBN, $\angle$CFS=$\angle$FCP, consequently BN $\parallel$ CP and $\angle$CBN=$\angle$BCP.

11. Consider the aggregate of the point A and *all* points such that for any one of them B, when BN $\parallel$ AM, also $\angle$ABN=$\angle$BAM, and designate this aggregate by F; and call L the intersection of F with any plane drawn through the straight AM.

F has a point, and one only, on every straight $\parallel$ AM; and L is divided by AM into two congruent parts.

Call the ray AM *the axis of L.* Evidently, in any one plane passing through the straight AM, there is for the axis ray AM a single line L. Call every line L so defined, the L of ray AM (in the plane, of course, that one considers). By the revolution of L around the straight AM we generate the F of which ray AM is called the axis, and which is, reciprocally, *the F of the axis ray AM.*

12. If B is any point of the L of ray AM, and BN $\parallel$ AM and $\angle$ABN=$\angle$BAM (§ 11), then the L of ray AM and the L of ray BN *coincide.* For suppose L′ the L of ray BN. Let C be any point of L′, and CP $\parallel$ BN and $\angle$BCP=$\angle$CBN (§ 11). Since BN $\parallel$ AM and $\angle$ABN=$\angle$BAM, therefore also CP $\parallel$ AM and $\angle$ACP=$\angle$CAM (§ 10). Consequently, C will be situated on L. And if C is any point of L, and CP $\parallel$ AM and $\angle$ACP=$\angle$CAM, then also CP $\parallel$ BN and $\angle$BCP=$\angle$CBN (§ 10); therefore C is likewise situated on L′ (§ 11). Thus L and L′ are identical, and every ray BN ($\parallel$ AM) is a new axis of L, and, if its origin is joined with that of any other axis, they make equal angles with the joining sect.

The same property may be demonstrated in the same manner for the surface F.

13. If BN ∥ AM, and CP ∥ DQ, and ∠BAM+∠ABN= st. ∠, then also ∠DCP+∠CDQ=st. ∠.

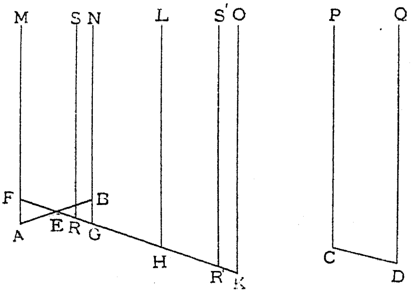


FIG. 11.

Let sect EA=sect EB, and ∠EFM=∠DCP (§ 4). Since ∠BAM + ∠ABN= st. ∠=∠ABN+∠ABG, we have ∠EBG=∠EAF.

If therefore we have in addition sect BG=sect AF, then △EBG ≌ △EAF, ∠BEG=∠AEF and G will fall on the ray FE. We have moreover ∠GFM + ∠FGN=st. ∠ (since ∠EGB=∠EFA).

Moreover GN∥FM (§ 6).

Therefore if MFRS ≌ PCDQ, then RS∥GN (§ 7), and R falls within or without the sect FG (unless sect CD=sect FG, in which case the proposition would be evident).

1. In the first case ∠FRS is not > st. ∠—∠RFM=∠FGN, since RS∥FM. But as RS∥GN, ∠FRS is not <∠FGN. Therefore ∠FRS=∠FGN, and ∠RFM + ∠FRS=∠GFM+ ∠FGN=st. ∠. Therefore also ∠DCP+∠CDQ=st. ∠.

2. If R falls without the sect FG, then ∠NGR=∠MFR.

Make MFGN ≌ NGHL ≌ LHKO, and so on until FK=FR or begins to be greater than FR. Then KO∥HL∥FM (§ 7).

If K falls on R then KO falls on RS (§ 1), and consequently ∠RFM+∠FRS=∠KFM+∠FKO=∠KFM+∠FGN=st. ∠. But if R falls within the sect HK, then (as in 1) we have ∠RHL+∠HRS=st. ∠=∠RFM+∠FRS=∠DCP+∠CDQ.

14. If BN∥AM, and CP∥DQ, and ∠BAM + ∠ABN< st. ∠, then also ∠DCP+∠CDQ<st. ∠.

Because, if ∠DCP + ∠CDQ were not <st. ∠, this sum (§ 1)

would be =st. ∠. Then we should have (§ 13) ∠BAM+∠ABN=st. ∠, which is contrary to the hypothesis.

15. In consideration of what has been established in §§13 and 14, *we will designate by* Σ *the system of geometry which rests on the hypothesis of the truth of Euclid's axiom* XI, *and by* S *the system founded on the contrary hypothesis.*

All results enunciated without designating expressly whether they belong to the system Σ *or the system* S, *should be considered as enunciated absolutely, that is true whether placed in system* Σ *or system* S.

16. If AM is the axis of a line L, this line L, in the system Σ, will be a straight ⊥ AM.

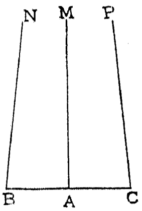


FIG. 12.

Suppose BN an axis at any point B of L; then in Σ, ∠BAM+∠ABN=st. ∠, therefore ∠BAM =rt. ∠.

And if C is any point of the straight AB, and CP‖AM, then (§ 13) ∠ACP=∠CAM, and consequently C will be on L (§ 11).

But in S, there exists nowhere on L nor on F three points in a straight. For some one of the axes AM, BN, CP, (e. g. AM) falls between the others, and then (§ 14) ∠BAM and ∠CAM are each < rt. ∠.

17. L in S is a line, and F a surface. For (§ 11) every plane drawn perpendicular to the axis ray AM through any point of F, cuts F in [the circumference of] a circle, of which the plane (§ 14) is perpendicular to no other axis BN. If we revolve F about BN, any point of F (§ 12) will remain on F, and the section of F by a plane not ⊥ ray BN will describe a surface. Now, whatever be the points A, B taken on F, F can be so moved *in its trace* that A falls upon the trace of B (§ 12).

Thus F is a uniform surface, a surface which will slide in its own trace.

It follows (§§ 11 and 12) that L is a uniform line, a line which will slide on its trace.*

18. The intersection of F with any plane drawn through a point A of F obliquely to the axis AM, is, in the system S, a circle.

Take A, B, C, three points of this section, and BN, CP, axes.

AMBN and AMCP make an angle, otherwise the plane determined by A, B, C, (§ 16), would contain AM, which is contrary to the hypothesis. Therefore the planes ⊥ the sects AB, AC at their mid points intersect (§ 10) in a certain axis ray FS of F, and we have FB=FA=FC.

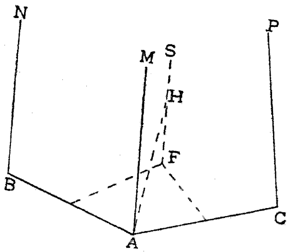


FIG. 13.

Make AH ⊥ FS, and revolve FAH around FS; A will describe a circle of radius HA, passing through B and C, and situated both in F and in the plane ABC; moreover, F and plane ABC have nothing common but the circle (○) HA (§ 16).

It is also evident that in revolving the portion FA of the line L (as radius) in F around A, its extremity will describe the circle with radius HA, ○ HA.

19. The perpendicular BT to the axis BN of L (drawn in the plane of L) is, in the system S, the tangent to the line L.

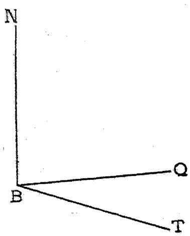


FIG. 14.

For L has in common with ray BT only the point B (§ 14). But if BQ is situated in the plane TBN, then the center of the section made in the F of ray BN by the plane drawn through BQ perpendicular to TBN (§ 18), is evidently on ray BQ; and if sect BQ is a diameter, it is clear that ray BQ will cut in Q the L of ray BN.

*It is not necessary to restrict the demonstration to the system S; we may easily establish that it is true absolutely for S and for Σ.

20. Any two points of F determine a line L (§§ 11 and 18); and since (§§ 16 and 19) L is $\perp$ to all its axes, every $\angle$ of lines L in F is equal to the $\angle$ of the planes drawn through its sides perpendicular to F.

21. Two line-rays, L-ray AP and L-ray BD, in the same surface F, making with a third line L, namely with line AB, interior angles of which the sun is $<$ st. $\angle$, intersect.

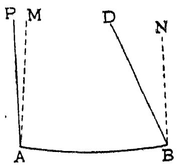


FIG. 15.

We shall designate by line AP, in F, the line L drawn through A and P, and by L-ray AP that half of this line beginning at A, which contains the point P.

Now, if AM, BN are axes of F, the hemi-planes AMP, BND intersect (§ 9), and F will meet their intersection (§§ 7 and 11). Therefore, L-ray AP and L-ray BD intersect.

From this it follows that Euclid's Axiom *XI* and all the consequences deduced from it in geometry and plane trigonometry are true absolutely in F, the lines L playing the role of straights. Consequently the trigonometric functions will be taken here in the same sense as in the system Σ; and the circle traced in F and having for radius a piece of line L equal to r, will have for length $2\pi r$; and area of $\odot$r (in F) $= \pi r^2$ (π designating the length of $\frac{1}{2}\bigcirc 1$ in F, that is to say, the known number 3.1415926+).

22. Let line AB be the L of ray AM, and C a point of ray AM. Suppose the $\angle$CAB (formed by the ray AM and the L-ray AB), translated first along the L-ray AB, then along the L-ray BA, each way to infinity. The path CD of the point C will be the line L of ray CM.

For, calling this latter L′, let D be any point of line CD, let DN be $\parallel$ CM, and B the point of L situated on the straight DN. We shall have BN $\parallel$ AM, and $\angle$ABN$=\angle$BAM, and sect

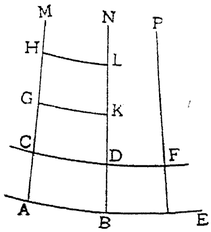


FIG. 16.

AC=sect BD, and consequently DN || CM and ∠CDN=∠DCM; therefore D is on L′. Moreover, if D is on L′ and if DN || CM, and B the point of L on the straight DN, we shall have AM || BN, and ∠BAM=∠ABN, and CM || DN and ∠DCM=∠CDN, whence follows that sect BD=sect AC, and D falls on the path of the point C. Therefore, L′ is identical with the line CD. We shall represent the relation of such a line L′ with L by the notation L′|||L.

23. If the line L represented by CDF is || ABE (§22); if, moreover, AB=BE, and the rays AM, BN, EP are axes, we shall evidently have CD=DF.

If A, B, E are any three points of line AB, and we have AB=n.CD, we shall also have AE=n. CF, and consequently (extending evidently to the case of AB, AE, DC incommensurable), AB:CD=AE:CF. The ratio AB:CD is, therefore, *independent of AB, and completely determined by AC.*

We shall designate the value of this ratio AB:CD by the capital letter (as X) corresponding to the small Italic (as x) by which we represent the sect AC.

24. Whatever be x and y, (§ 23), $Y=X^{\frac{y}{x}}$.

For, one of the quantities x, y is a multiple of the other (e. g. y is a multiple of x) or it is not.

If y=n.x, take x=AC=CG=GH=&c., until we get AH=y.

Moreover, take CD || GK || HL.

We have (§ 23) X=AB:CD=CD:GK=GK:HL, and consequently

$$\frac{AB}{HL}=\left(\frac{AB}{CD}\right)^{n},$$

or $Y=X^n=X^{\frac{y}{x}}$.

If x, y are multiples of i, we shall have in accordance with the above, $X=I^m$, $Y=I^n$, and consequently

$$Y=X^{\frac{n}{m}}=X^{\frac{y}{x}}.$$

This conclusion is easily extended to the case where x and y are incommensurable.

If $q=y-x$, then $Q=Y:X$.

In the system Σ, for every value x, we have $X=1$.

In the system S, on the contrary, $X>1$, and for any values of AB and ABE there is a line ∥ AB such that CDF=AB, whence results AMBN $\cong$ AMEP, though the first of these two figures may be any multiple of the second; a singular result, but evidently not showing any absurdity in the system S.

25. *In every rectilineal triangle, the circles with radii equal to its sides are to each other as the sines of the opposite angles.*

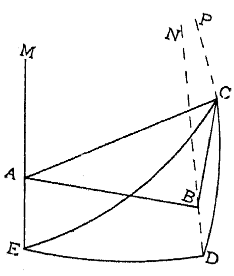


FIG. 17.

Take ∠ABC=rt. ∠, and AM ⊥ BAC, and BN and CP ∥ AM.

We shall have CAB ⊥ AMBN, and consequently (since CB ⊥ BA), CB ⊥ AMBN; therefore, CPBN ⊥ AMBN. Suppose that the F of ray CP cuts the straights BN, AM respectively in D and E, and the bandes CPBN, CPAM, BNAM along the L-lines CD, CE, DE. Then (§ 20) ∠CDE will be equal to the angle of NDC, NDE, and hence=rt. ∠; we have in the same way ∠CED=∠CAB. Now, (§ 21) in △CED formed by the L-lines, (supposing always here the radius=1), we have

$$EC:DC=1:\sin DEC=1:\sin CAB.$$

We have also (§ 21)

$$EC:DC=\bigcirc.EC:\bigcirc.DC \text{ (in F)}=\bigcirc.AC:\bigcirc.BC \text{ (§ 18)}.$$

Consequently we conclude

$$\bigcirc.AC:\bigcirc.BC=1:\sin CAB,$$

whence it follows that the theorem enunciated is established for any triangle.

26. *In any spherical triangle, the sines of the sides are to each other as the sines of the angles opposite.*

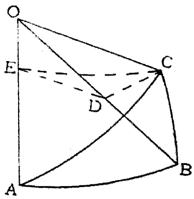


FIG. 18.

Take $\angle$ABC=rt. $\angle$, and CED $\perp$ to the radius OA of the sphere. We shall have CED $\perp$ AOB, and (BOC being also $\perp$ to BOA), CD $\perp$ OB. Now, in the triangles CEO, CDO, we have (§ 25)
$\bigcirc$.EC:$\bigcirc$.OC:$\bigcirc$.DC=sin COE: 1 : sin COD =sin AC: 1 : sin BC.

But we have also (§ 25) $\bigcirc$.EC:$\bigcirc$.DC=sin CDE: sin CED. Therefore, sin AC: sin BC=sin CDE: sin CED. But CDE= rt. $\angle$=CBA, and CED=CAB. Consequently,

sin AC: sin BC=1 : sin A.

From this follows the whole of spherical trigonometry, which is thus established independently of Euclid's Axiom XI.

27. If AC and BD are $\perp$ AB and we translate the $\angle$CAB along the ray AB, we shall have, designating by CD the path described by the point C,

CD: AB=sin u: sin v.

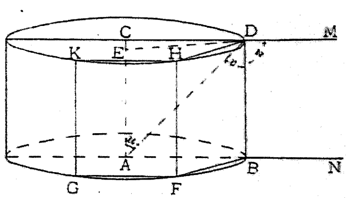


FIG. 19.

Take DE $\perp$ CA. In the triangles ADE, ADB, we have (§ 25)

$\bigcirc$.ED:$\bigcirc$.AD:$\bigcirc$.AB= sin u: 1 : sin v.

In revolving BACD around AC, the point B will describe $\bigcirc$.AB, and the point D will describe $\bigcirc$.ED.

Designate here by $\odot$.CD the path of the line CD. Moreover, let there be any polygon BFG . . . inscribed in $\bigcirc$.AB.

Passing through all the sides BF, FG . . . . planes $\perp$ $\odot$.AB we form thus a polygonal figure of the same number of sides in ⊙.CD, and we may demonstrate, as in § 23, that CD : AB = DH : BF = HK : FG = . . . , and consequently

$$DH + HK + \ldots : BF + FG + \ldots = CD : AB.$$

If we make each of the sides BF, FG . . . approach the limit zero, we have

$$BF + FG + \ldots \doteq \bigcirc.AB \quad \text{and}$$

$$DH + HK + \ldots \doteq \bigcirc.ED.$$ We have therefore also ○.ED : ○.AB = CD : AB. Now, we already had ○.ED : ○.AB = sin u : sin v. Consequently,

$$CD : AB = \sin u : \sin v.$$

If AC goes away from BD to infinity, then the ratio CD : AB, and consequently also the ratio sin u : sin v remains constant. Now $u \doteq$ rt. ∠ (§ 1), and if DM ∥ BN, $v \doteq z$. Therefore, CD : AB = 1 : sin z.

We shall designate this path CD by CD ⫴ AB.

28. If BN ∥ AM, and ∠ABN = ∠BAM, and C a point of ray AM, then putting AC = x (§ 23) we shall have

$$X = \sin u : \sin v.$$

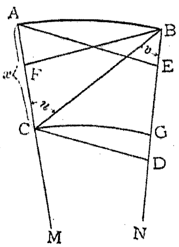


FIG. 20.

For, CD and AE being ⊥ BN, and BF ⊥ AM, we shall have (as in § 27)

$$\bigcirc.BF : \bigcirc.CD = \sin u : \sin v.$$

Now evidently BF = AE. Therefore

$$\bigcirc.EA : \bigcirc.DC = \sin u : \sin v.$$

But in the F-surfaces of AM and CM, which cut AMBN along AB and CG, we have (§ 21)

$$\bigcirc.EA : \bigcirc.DC = AB : CG = X.$$

Therefore also

$$X = \sin u : \sin v.$$

29. If $\angle$BAM=rt. $\angle$, and sect AB=y, and BN $\parallel$ AM, we shall have in the system S,

$$\mathrm{Y}=\cot \tfrac{1}{2}\, u.$$

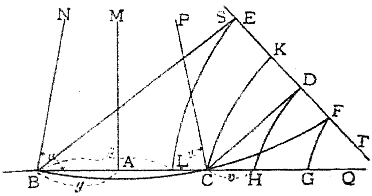


FIG. 21.

For, if we suppose sect AB= sect AC, and CP $\parallel$ AM (and so BN $\parallel$ CP and $\angle$CBN= $\angle$BCP), and $\angle$PCD=$\angle$QCD, then we can draw (§ 19) DS $\perp$ ray CD so that DS $\parallel$ CP, and consequently (§ 1) DT $\parallel$ CQ. Moreover, if BE $\perp$ ray DS, then (§ 7) DS $\parallel$ BN, consequently (§ 6) BN $\parallel$ ES, and (since DT $\parallel$ CG) BQ $\parallel$ ET. Therefore, (§ 1) $\angle$EBN=$\angle$EBQ. Let BCF be an L-line of BN, and FG, DH, CK, EL, L-lines of FT, DT, CQ, &c. We shall have (§ 22) HG=DF=DK=HC; therefore,

$$\mathrm{CG}=2\mathrm{CH}=2\, v.$$

In the same way BG=2BL=$2z$.

Now BC=BG—GC; so $y=z-v$, whence (§ 24) Y=Z:V.

Finally we have (§ 28)

$$\mathrm{Z}=1:\sin \tfrac{1}{2}\, u,$$

$$\mathrm{V}=1:\sin\,(\text{rt. } \angle - \tfrac{1}{2}\, u).$$

Therefore, $\mathrm{Y}=\cot \tfrac{1}{2}\, u.$

30. It is easy to see (after (§ 25) that solution of the problem of Plane Trigonometry, in the system S, requires the expression of the circle in terms of the radius. Now, we are able to obtain this by the rectification of the line L.

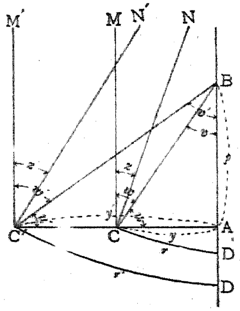


FIG. 22.

Let AB, CM, C′M′ be straights $\perp$ ray AC, and B any point of ray AB. We shall have (§ 25)

$$\sin u : \sin v = \bigcirc p : \bigcirc y,$$

$$\sin u' : \sin v' = \bigcirc p : \bigcirc y';$$

Consequently, $\frac{\sin u}{\sin v}.\bigcirc y=\frac{\sin u'}{\sin v'}.\bigcirc y'$.

Now, we have (§ 27) $\sin v:\sin v'=\cos u:\cos u'$.

Therefore, $\frac{\sin u}{\cos u}.\bigcirc y=\frac{\sin u'}{\cos u'}.\bigcirc y'$.

or $\bigcirc y:\bigcirc y'=\tan u':\tan u=\tan w:\tan w'$.

Take now CN and C′N′ ∥ AB, and CD, C′D′ L-lines ⊥ AB. We shall have then (§ 21)

$\bigcirc y:\bigcirc y'=r:r'$, whence

$r:r'=\tan w:\tan w'$.

Make p increase from A to infinity; then $w \doteq z$ and $w' \doteq z'$, whence results also $r:r'=\tan z:\tan z'$.

Designate by i the *constant* ratio

$r:\tan z$ (*independent* of r).

If we suppose $y \doteq 0$, then

$\frac{r}{y}=\frac{i \tan z}{y} \doteq 1$, and consequently

$\frac{y}{\tan z} \doteq i$. From §29, it follows that $\tan z=\frac{1}{2}(\mathrm{Y}-\mathrm{Y}^{-1})$.

Therefore $\frac{2y}{\mathrm{Y}-\mathrm{Y}^{-1}} \doteq i$,

or (§ 24) $\frac{2y.\mathrm{I}^{\frac{y}{i}}}{\mathrm{I}^{\frac{2y}{z}}-1} \doteq i$.

Now, we know that the limit of this expression, for

$y \doteq o$, is $\frac{i}{\text{nat. log I}}$. Therefore,

$\frac{i}{\text{nat. log I}}=i$, and consequently

$\mathrm{I}=e=2.7182818+$,

a number which presents itself here in a remarkable manner.

Designating henceforth by i the sect of which the $\mathrm{I}=e$, we shall have

$$r = i \tan z.$$

We have found elsewhere (§ 21) $\bigcirc y = 2\pi r$.

Therefore,

$$\bigcirc y = 2\pi i \tan z = \pi \mathrm{i} \left(\mathrm{Y} - \mathrm{Y}^{-1}\right) = \pi i \left(e^{\frac{y}{\mathrm{i}}} - e^{-\frac{y}{\mathrm{i}}}\right)$$

$$= \frac{\pi y}{\text{nat. log Y}} \left(\mathrm{Y} - \mathrm{Y}^{-1}\right) \quad (§\ 24).$$

31. For the trigonometric solution of all right-angled rectilineal triangles (whence is easily deduced that of all rectilineal triangles whatsoever), in the system S, three equations suffice. Let c be the hypothenuse, a, b the sides of the right angle, and α, β the angles respectively opposite to a and b. These three equations shall be those which express relations.

I. Between a, c, α;

II. Between a, α, β;

III. Between a, b, c.

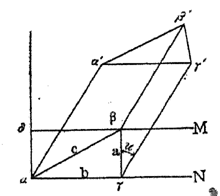


FIG. 23.

From these equations we shall deduce afterward three others by elimination.

I. From §§ 25 and 30 we get

$$1 : \sin \alpha = \left(\mathrm{C} - \mathrm{C}^{-1}\right) : \left(\mathrm{A} - \mathrm{A}^{-1}\right) =$$

$$= \left(e^{\frac{c}{\mathrm{i}}} - e^{-\frac{c}{\mathrm{i}}}\right) : \left(e^{\frac{a}{\mathrm{i}}} - e^{-\frac{a}{\mathrm{i}}}\right),$$ *an equation between* c, a, *and* α.

II. From §27 we deduce (BM being $\parallel \gamma$n)

$$\cos \alpha : \sin \beta = 1 : \sin u.$$

Now, we have (§ 29)

$1 : \sin u = \frac{1}{2}\left(\mathrm{A} + \mathrm{A}^{-1}\right)$; therefore $\cos \alpha : \sin \beta = \frac{1}{2}\left(\mathrm{A} + \mathrm{A}^{-1}\right) =$

$$\frac{1}{2}\left(e^{\frac{a}{\mathrm{i}}} + e^{-\frac{a}{\mathrm{i}}}\right),$$ *an equation between* α, β, *and* a.

III. Take $\alpha\alpha' \perp \beta\alpha\gamma$; $\beta\beta'$ and $\gamma\gamma' \parallel \alpha\alpha'$ (§ 27), and $\beta'\alpha'\gamma' \perp \alpha\alpha'$. We will evidently have (as in § 27)

$$\frac{\beta\beta'}{\gamma\gamma'}=\frac{1}{\sin u}=\tfrac{1}{2}(A+A^{-1}),$$

$$\frac{\gamma\gamma'}{\alpha\alpha'}=\tfrac{1}{2}(B+B^{-1}),$$

$$\frac{\beta\beta'}{\alpha\alpha'}=\tfrac{1}{2}(C+C^{-1}).\quad \text{Consequently}$$

$$\tfrac{1}{2}(C+C^{-1})=\tfrac{1}{2}(A+A^{-1}).\tfrac{1}{2}(B+B^{-1}),\quad \text{or}$$

$$e^{\frac{c}{i}}+e^{-\frac{c}{i}}=\tfrac{1}{2}\left(e^{\frac{a}{i}}+e^{-\frac{a}{i}}\right)\left(e^{\frac{b}{i}}+e^{-\frac{b}{i}}\right),$$

an equation between a, b, and c.

If $\gamma\alpha\delta$=rt. $\angle$., and we have $\beta\delta \perp \alpha\delta$, then we shall get

$$\bigcirc c : \bigcirc a = 1 : \sin \alpha, \text{ and}$$

$$\bigcirc c : \bigcirc (d = \beta\delta) = 1 : \cos \alpha.$$

Therefore, designating by $\bigcirc x^2$, for any value of x, the product $\bigcirc x . \bigcirc x$, we shall evidently have

$$\bigcirc a^2 + \bigcirc d^2 = \bigcirc c^2.$$

Now, we have found (§ 27 and § 31, II)

$$\bigcirc d = \bigcirc b . \tfrac{1}{2}(A+A^{-1}),\quad \text{Consequently}$$

$$\left(e^{\frac{c}{i}}-e^{-\frac{c}{i}}\right)^2=\tfrac{1}{4}\left(e^{\frac{a}{i}}+e^{-\frac{a}{i}}\right)^2 \cdot \left(e^{\frac{b}{i}}-e^{-\frac{b}{i}}\right)^2+\left(e^{\frac{a}{i}}-e^{-\frac{a}{i}}\right)^2,$$

another relation between a, b, and c, the second member of which may be easily put into a form symmetric or invariable.

Finally, from the equations

$$\frac{\cos\alpha}{\sin\beta}=\tfrac{1}{2}(A+A^{-1}),\ \frac{\cos\beta}{\sin\alpha}=\tfrac{1}{2}(B+B^{-1}), \text{ we get (after II)}$$

$$\cot\alpha\ \cot\beta=\tfrac{1}{2}\left(e^{\frac{c}{i}}+e^{-\frac{c}{i}}\right),$$

an equation between α, β, and c.

32. It still remains to show briefly the means of resolving problems in the system S. After having expounded this in regard to the most ordinary examples, we shall see finally what this theory is able to give.

I. Take AB a line in a plane, and $y=f(x)$ its equation in rectangular coordinates. Designate by dz any increment of z, and by dx, dy, du the increments of x, of y, and of the area u, corresponding to this increment dz. Take BH $|||$ CF; express (§31) $\frac{\mathrm{BH}}{dx}$ by means of y, and seek the *limit* of $\frac{dx}{dy}$, when dx tends toward the limit zero (which is always understood when one seeks such limits.

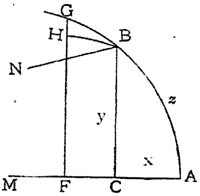


FIG. 24.

We shall then know the limit of $\frac{dy}{\mathrm{BH}}$, and so tan HBG; and consequently (since evidently HBC can be neither $>$, nor $<$rt. $\angle$, and so is$=$rt. $\angle$), the *tangent* at B of the line BG will be determined by means of y.

II. We can demonstrate that

$$\frac{dz^2}{dy^2+\overline{\mathrm{BH}}^2} \doteq 1.$$

Thence we deduce the limit of $\frac{dz}{dx}$, and from it we get, by integration, the expression for z in terms of x.

Given any real curve, we can find its equation in the system S.

For example, to find the equation of a line L. Let ray AM be the axis of the line L; every straight drawn through A, other than the straight AM, meeting L (§ 19), the random ray CB, starting from a point of ray AM, will meet L.

Now, if BN is an axis, we have

$$\mathrm{X}=1:\sin \mathrm{CBN} \ (\S\,28),$$

$$\mathrm{Y}=\cot \tfrac{1}{2}\,\mathrm{CBN} \ (\S\,29), \text{ whence}$$

we get $\mathrm{Y}=\mathrm{X}+\sqrt{\mathrm{X}^2-1}$,

or

$$e^{\frac{y}{i}}=e^{\frac{x}{i}}+\sqrt{e^{\frac{2x}{i}}-1},$$

which is the equation sought.

Hence we get

$$\frac{dy}{dx} \doteq \mathrm{X}.(\mathrm{X}^2-1)^{-\frac{1}{2}}$$

Now, $\frac{\mathrm{BH}}{dx}=1:\sin \mathrm{CBN}=\mathrm{X}$. Therefore

$$\frac{dy}{\mathrm{BH}} \doteq (\mathrm{X}^2-1)^{-\frac{1}{2}}$$

$$1+\frac{dy^2}{\mathrm{BH}^2} \doteq \mathrm{X}^2.(\mathrm{X}^2-1)^{-1},$$

$$\frac{dz^2}{\mathrm{BH}^2} \doteq \mathrm{X}^2.(\mathrm{X}^2-1)^{-1},$$

$$\frac{dz}{\mathrm{BH}} \doteq \mathrm{X}.(\mathrm{X}^2-1)^{-\frac{1}{2}}, \quad \text{and}$$

$$\frac{dz}{dx} \doteq \mathrm{X}^2.(\mathrm{X}^2-1)^{-\frac{1}{2}}, \quad \text{whence,}$$

integrating, we get (as in §30)

$$z=\mathrm{i}(\mathrm{X}^2-1)^{\frac{1}{2}}=\mathrm{i}\cot \mathrm{CBN}.$$

III. Evidently

$$\frac{du}{dx} \doteq \frac{\mathrm{HFCBH}}{dx}.$$

If this quantity is not given in y, it is necessary to express it in terms of y, and then we get u from it by integration.

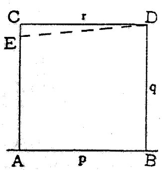


FIG. 25.

Putting $\mathrm{AB}=p$, $\mathrm{AC}=q$, $\mathrm{CD}=r$, and $\mathrm{CABD}=\mathrm{S}$, we might show (as in II) that

$$\frac{ds}{dq} \doteq r, \text{ a quantity equal to}$$

$$\tfrac{1}{2}p\left(e^{\frac{q}{\mathrm{i}}}+e^{-\frac{q}{\mathrm{i}}}\right), \text{ whence, integrating,}$$

$$\mathrm{S}=\tfrac{1}{2}p\mathrm{i}\left(e^{\frac{q}{\mathrm{i}}}-e^{-\frac{q}{\mathrm{i}}}\right).$$

We might also obtain this result without integration.

For example, if we establish the equation of the circle (after §31, III), of the straight, (from §31, II), of a conic (from what just precedes), we could express also the areas bounded by these lines.

We know that a surface t, ||| a plane figure p (at the distance q) is to p in the ratio of the second powers of homologous lines, that is to say in the ratio of

$$\tfrac{1}{4}\left(e^{\frac{q}{\mathrm{i}}}+e^{-\frac{q}{\mathrm{i}}}\right)^2 : 1.$$

It is easy to see, moreover, that the calculation of volume, treated in the same manner, requires two integrations (the differential itself being determinable only by integration).

It is necessary first of all to investigate the volume contained between p and t, and the aggregate of all the straights $\perp p$ and joining the boundaries of p and t.

We find for the volume of this solid (whether by means of integration or otherwise)

$$\tfrac{1}{8}\,p\mathrm{i}\left(e^{\frac{2q}{\mathrm{i}}}-e^{-\frac{2q}{\mathrm{i}}}\right)+\tfrac{1}{2}\,pq.$$

The surfaces of bodies may also be calculated in the system S, as well as the *curvatures*, the involutes, the evolutes of any lines, etc.

As to curvature, in the system S, either it will be the curvature of the line L itself, or we may determine it either by the radius of a circle, or by the *distance* of a straight from the curve ||| to this straight; and it is easy to make it evident, after what precedes, that there is not, in a plane any uniform line other than L-lines, circles, and the curves ||| to straights.

IV. For the circle we have (as in III) $\dfrac{d\odot x}{dx} = \bigcirc x$, whence (§ 29), integrating, we get

$$\odot x=\pi \mathrm{i}^2\left(e^{\frac{x}{\mathrm{i}}}-2+e^{-\frac{x}{\mathrm{i}}}\right).$$

V. Take u=CABDC the area comprised between an L-line, AB=r, a ||| to that line, CD=y, and the sects AC=BD=x.

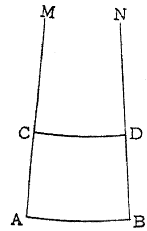


FIG. 26.

We have $\frac{du}{dx} \doteq y$, and (§ 24) $y = re^{-\frac{x}{i}}$, whence

integrating $u = ri\left(1 - e^{-\frac{x}{i}}\right)$.

If x increases to infinity, then, in the system S, $e^{-\frac{x}{i}} \doteq 0$, and consequently $u \doteq ri$. We shall call this limit the *size* of MABN.

We may see in the same manner that, if p is a figure traced on F, the space comprised between p and the aggregate of axes drawn through the different points of the boundary of p is equal to $\frac{1}{2}pi$.

Let $2u$ be the angle at the center of the spherical calotte z, and p a great circle, and x the arc FC corresponding to the angle u. We shall have (§ 25)

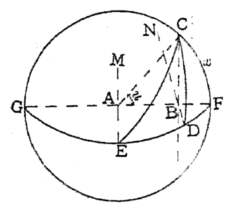


FIG. 27.

$1 : \sin u = p : \bigcirc.\mathrm{BC}$, whence

$\bigcirc.\mathrm{BC} = p \sin u$. We have, besides,

$$x = \frac{pu}{2\pi}, \qquad dx = \frac{pdu}{2\pi}.$$

Moreover, $\frac{dz}{dx} \doteq \bigcirc.\mathrm{BC}$,

$$\frac{dz}{du} \doteq \frac{p^2}{2\pi} \sin u, \text{ and, integrating,}$$

$$z = \frac{\text{ver sin } u}{2\pi} p^2.$$

Imagine the surface F on which is situated the circle p (passing through the middle F of the calotte). Draw through AF and AC the hemi-planes FEM, CEM, perpendicular to F and cutting F along FEG and CE; and consider the L-line CD (drawn through C perpendicular to FEG), and the L-line CF.

We shall have (§ 20) CEF$=u$, and (§ 21)

$$\frac{fd}{p}=\frac{\text{ver sin } u}{2\pi}, \text{ whence } z=\text{FD}.p.$$

Now (§ 21) $p=\pi$.FDG; therefore

$z=\pi$.FD.FDG. But (§ 21)

FD.FDG=FC.FC; consequently

$z=\pi$.FC.FC$=\odot$.FC, in F.

Now let BJ=CJ$=r$; we shall have (§ 30) $2r=i(\text{Y}-\text{Y}^{-1})$, whence, (§ 21)

$$\odot 2r(\text{in F})=\pi i^2(\text{Y}-\text{Y}^{-1})^2.$$

We also have (*IV*)

$$\odot 2y=\pi i^2(\text{Y}^2-2+\text{Y}^{-2}).$$

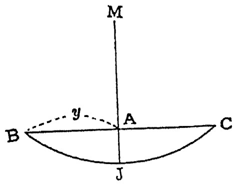


Fig. 28.

Therefore, $\odot 2r$ (in F)$=\odot 2y$, and consequently *the surface* z *of the segment of a sphere is equal to the surface of the circle described with the chord* fc *as radius.*

Therefore the whole sphere has for surface

$$\odot.\text{FG}=\text{FDG}.p=\frac{p^2}{\pi},$$

and the surfaces of spheres are to each other as the second powers of their great circles.

VII. We find in like manner that, in the system S, the volume of the sphere of radius x is equal to

$$\tfrac{1}{2}\pi i^2(\text{X}^2-\text{X}^{-2})-2\pi i^2 x.$$

The surface generated by the revolution of the line CD around AB is equal to

$$\tfrac{1}{2}\pi i p(\text{Q}^2-\text{Q}^{-2}),$$

and the solid generated by CABDC is equal to

$$\tfrac{1}{4}\pi i p(\text{Q}-\text{Q}^{-1})^2.$$

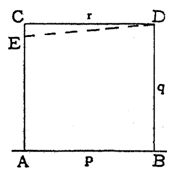


Fig. 29.

We suppose, for the sake of brevity, the method by which one may obtain without integration all the results reached from IV thus far.

We can demonstrate that *the limit of every expression containing the letter i* (and consequently founded on the hypothesis that a magnitude i exists), *when i increases to infinity, gives the corresponding expression in the system Σ* (and consequently under the hypothesis that a magnitude i does not exist), provided that we do not meet identical equations.

But we must be very careful not to get the idea that the system itself may be changed at will (for it is entirely determined in itself and by itself); it is only *the hypothesis* which may vary, and which we may change successively, so far as we are not conducted to an absurdity. In *supposing* therefore that, in such an expression, the letter i, in case the system S is that of reality, designates the unique quantity of which the I has e for its value, if we come to recognize that it is the system Σ, which is really actual, we conceive that the limit in question is to be taken in place of the primitive expression. Then it is evident that with this understanding, *all the expressions founded on the hypothesis of the reality of the system S will be true absolutely, even when we are completely ignorant whether or not the system Σ is the system of reality.*

So, for example, from the expression obtained in §30 we easily get (either by means of differentiation or otherwise) the known value in the system Σ,

$$\bigcirc x = 2\pi x.$$

From I (§31) we conclude, by suitable transformations,

$$1 : \sin \alpha = c : a;$$

from II we get

$$\frac{\cos \alpha}{\sin \beta} = 1, \text{ and consequently}$$

$$\alpha + \beta = 1.$$

The first equation of III becomes identical, and so it is true in the system Σ, although it there determines nothing. From the second we conclude

$$c^2 = a^2 + b^2.$$

These are the known fundamental equations of plane trigonometry in the system Σ.

Moreover, we find (after §32) in the system Σ, for the area and the volume in III the same value pq.

We have, from IV,

$$\odot x = \pi x^2.$$

According to VII, the globe of radius x is

$$= \tfrac{4}{3}\pi x^3, \text{ etc,}$$

The theorems enunciated at the end of VI are evidently *true without conditions.*

33. It still remains to set forth (as we promised in §32) what is the end of this theory.

I. Is it the system Σ or the system S which exists in reality?

That is what we cannot decide.

II. All the results deduced from the falsity of Axiom *XI* (always taking these words in the sense of §32) are *absolutely true*, and in this sense, *depend on no hypothesis.*

There is therefore *a plane trigonometry a priori, in which the system alone really remains unknown;* and where we lack only the *absolute* magnitudes in the expressions, but where a single known case would evidently fix the whole system. On the contrary, spherical trigonometry is established absolutely in §26.

We have, on the surface F, a geometry wholly analogous to the plane geometry of the system Σ.

III. If it were established that it is the system Σ which exists, nothing more would remain to be known on this point.

But if it were *established* that the system Σ *does not exist*, then (§31), being given, for example, in a concrete manner, the sides x, y, and the rectilineal angle which they include, it is clear that it would be impossible in itself and by itself to solve absolutely the triangle, that is to say, to determine *a priori* the other angles and the ratios of the third side to the two given sides, unless one could determine the quantities X, Y. For that, it would be necessary to have in concrete form a certain sect a of which the A was known. Then i would be *the natural unit for length* (as e is the base of *natural* logarithms).

If the existence of this quantity i is supposed to be known, we see how one could construct it, at least with a high degree of approximation, for practical use.

IV. In the sense explained (I and II), we may evidently apply everywhere the modern analytic method (so useful when one employs it within suitable limits).

V. Finally, the reader will not be sorry to see that in case it is the system S, and not the system Σ, which really exists, we can construct a rectilineal figure equivalent to a circle.

34. Through D we may draw DM$\|$AN in the following manner. From the point D drop DB $\perp$ AN; at any point A of the straight AB erect AC $\perp$ AN (in the plane DBA) and let fall DC $\perp$ AC. We will have (§ 27) $\bigcirc$.CD:$\bigcirc$.AB$=$ 1 : sin z, provided that DM$\|$BN. Now

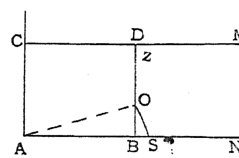


FIG. 30.

sin z is not $>$1; therefore AB is not $>$ DC. Therefore a quadrant described from the center A in BAC, with a radius $=$DC, will have a point B on $\bigcirc$ in common with ray BD. In the first case, we have evidently z=rt. $\angle$. In the second case we shall have (§25)

$$\bigcirc.\text{AO}(=\text{CD}):\bigcirc.\text{AB}=1:\sin\text{AOB}$$

and consequently z=AOB.

If therefore we take z=AOB, then DM will be $\|$ BN.

35. In the system S we may, as follows, draw a straight $\perp$ to one of the sides of an acute angle and at the same time $\|$ to the other side.

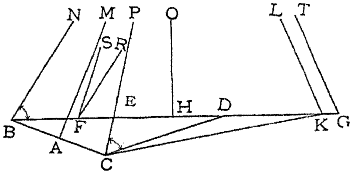


FIG. 31.

Take AM $\perp$ BC, and suppose AB=AC sufficiently small (§19) to make, when we draw BN$\|$AM (§ 34) ABN > the given angle.

Draw also CP$\|$AM (§34), and take NBG and PCB each equal to the given angle. Rays BG and CD will meet; because if ray BG (falling *by construction* within NBC) cuts ray CP in E, we shall have (since ∠CBN=∠BCP) ∠EBC∠ECB, and so EC< EB. Take EF=EC, EFR=ECD, and FS$\|$EP, then FS will fall within the angle BFR. Because, since BN$\|$CP, whence BN$\|$EP, and BN$\|$FS, we shall have (§14)

$$\angle FBN + \angle BFS < \text{st. } \angle = FBN + BFR.$$

Therefore, BFS< BFR. Consequently, ray FR cuts ray EP, and so ray CD also cuts ray EG somewhere in D. Take now DG=DC and DGF=DCP=GBN. We shall have (since ∠GCD=∠CGD) ∠GBN=∠BGT and ∠GCP=CGT. Let K (§19) be the point where the line L of BN meets the ray BG and KL the axis of this L-line. We shall have ∠KBN= ∠BKL, and so BKL=BGT=DCP.

Moreover, CKL=KCP. Therefore, evidently K falls on G, and GT$\|$BN. If now we erect HO $\perp$ BG at its mid point, we shall have constructed HO$\|$BN.

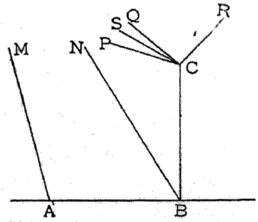


FIG. 32.

36. Having given the ray CP and the plane MAB, take CB $\perp$ the plane MAB, BN (in hemi-plane BCP) $\perp$ BC, and CQ$\|$BN (§34). The meeting of ray CP (if this ray falls within BCQ) with ray BN (in the plane CBN), and consequently with the plane MAB, may be determined, And if we are given the two planes PCQ, MAB, and we have CB $\perp$ to plane

MAB, CR ⊥ plane PCQ; and (in plane BCR), BN ⊥ BC, CS ⊥ CR, BN will fall in plane MAB, and CS in plane PCQ; and when we have found the intersection of the straight BN with the straight CS (if there is one), the perpendicular drawn through this intersection, in PCQ, to the straight CS will evidently be the tntersection of plane MAB and plane PCQ.

37. On the straight AM‖BN, there is a point A, such that ∠BAM=∠ABN.

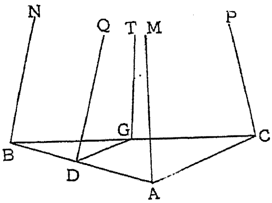


FIG. 33.

If (according to §34) we construct outside of the plane NBM, GT ‖ BN, and make BG ⊥ GT, GC=GB, and CP‖GT; and so draw the hemi-plane TGD that it makes with hemi-plane TGB an angle equal to that which hemi-plane PCA makes with PCB.

Seek then (§36) the intersection DQ of hemi-plane TGD with hemi-plane NBD, and finally draw BA ⊥ DQ.

We shall have, by reason of the similtude of the triangles of L lines traced on the F of BN (§21), DB=DA, and ∠BAM =∠ABN.

We readily conclude from this, that, L-lines being given by their extremities alone, we may obtain in this manner, in F, a fourth proportional, or a mean proportional, and execute, without recourse to Axiom XI, all the geometric constructions made on the plane in the system Σ. Thus, for example, we can geometrically divide a perigon into any special number of equal parts, if we know how to make this special partition in the system Σ.

38. If we construct (§37) for example, NBQ=$\frac{1}{3}$ rt. ∠, and draw (§35), in the system S, AM ⊥ ray BQ and ‖ BN; if we determine, again (§37), ∠BIM=∠IBN, we shall have, putting IA=x (§28), X= 1 : sin $\frac{1}{3}$ rt. ∠=2, and x will be constructed *geometrically*.

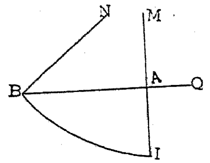


FIG. 34.

We may calculate NBQ so that IA differs from i as little as we choose, which happens for $\sin \text{NBQ} = \frac{1}{e}$.

39. If in a plane PQ and ST are ||| to the straight MN (§27), and the perpendiculars AB, CD are equal, we shall evidently have $\triangle$DEC $\cong$ $\triangle$BEA; consequently the angles (may be mixtilinear) ECP, EAT would coincide, and we have EC=EA. If, besides CF=AG, then $\triangle$ACF $\cong$ $\triangle$CAG, and each of them is the half of the *quadrilateral* FAGC.

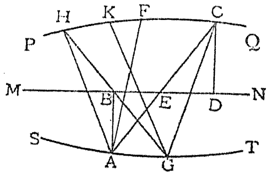


FIG. 35.

If FAGC, HAGK are two of these quadrilaterals, of base AG, contained between PQ and ST, we may demonstrate their equivalence (as in Euclid), as also the equivalence of the triangles AGC, AGH, on a common base AG, and having their vertices on PQ. Moreover, we have ACF=CAG, GCQ=CGA, and ACF+ACG+GCQ=st. $\angle$ (§32); consequently we also have CAG+ACG+CGA=st. $\angle$. Therefore, in every triangle ACG of this sort, the sum of the angles=st. $\angle$. Whether the straight AG coincides with AG (||| MN), or not, the equivalence of the triangles AGC, AGH, *as well in relation to their areas as in relation to the sum of their angles*, is evident.

40. *Equivalent triangles* ABC, ABD, (which we will henceforth suppose rectilineal), *having one side equal, have the sums of their angles equal.*

Draw MN through the mid points of AC and BC, and take (through the point C) PQ ||| MN. The point D will fall on PQ.

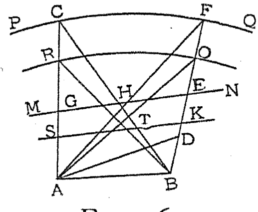


FIG. 36.

For, if ray BD cuts the straight MN at the point E, and and consequently PQ at F making EF=EB, we shall have

$\triangle$ABC=$\triangle$ABF, and so also $\triangle$ABD=$\triangle$ABF, whence it follows that D falls at F.

But if ray BD does not cut the straight MN, let C be the point where the perpendicular from the middle of AB meets PQ, and let GS=HT, so that the straight ST meets the ray BD prolonged in a certain point K (which is possible as we have seen in §4). Take also SR=SA, RO ||| ST, and O the intersection of ray BK with RO. We would then have $\triangle$ABR=$\triangle$ABO (§39), and consequently $\triangle$ABC>$\triangle$ABD, which would be contrary to the hypothesis.

41. *Equivalent triangles ABC, DEF have the sums of their angles equal.*

Draw MN through the mid points of AC and BC, and PQ through the mid points of DF and FE; and take RS ||| MN, TO ||| PQ.

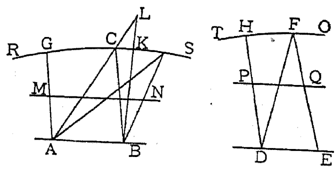


FIG. 37.

The perpendicular AG to RS will equal the perpendicular DH to TO, or will differ from it; for example, DH will be the greater.

In each of these two cases, the $\bigcirc$.DF, described from center A, will have with line-ray GS a common point K, and then (§39) we shall have $\triangle$ABK=$\triangle$ABC=$\triangle$DEF. Now the $\triangle$ABK (§40) has the same angle-sum as $\triangle$DEF, and (§39) the same angle-sum as $\triangle$ABC. Therefore the triangles ABC, DEF have each the same angle-sum.

In the system S the reverse of this theorem is true.

Take ABC, DEF two triangles having the same angle-sum, and $\triangle$BAL=$\triangle$DEF. These latter triangles will have, from what precedes, the same angle-sum, and consequently so will $\triangle$ABC and $\triangle$ABL.

From this follows evidently

$$BCL+BLC+CBL=\text{st. } \angle.$$

Now (§31) the angle-sum of every triangle, in the system S, is $<$ st. $\angle$.

Therefore L falls necessarily on C.

42. Let u be the supplement of the angle-sum of the $\triangle$ABC, and v the supplement of the angle sum of $\triangle$DEF. We shall have $\triangle ABC : \triangle DEF = u : v$.

Let p be the area of each of the equal triangles ACG, GCH, HCB, DFK, KFE, and let $\triangle ABC = m.p$, and $\triangle DEF = n.p$. Designate by s the angle-sum of any one of the triangles equivalent to p. We shall evidently have

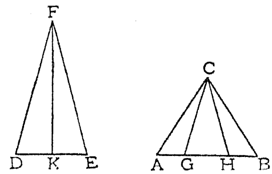


FIG. 38.

$$\text{st. } \angle - u = m.s - (m-1)\text{st. } \angle = \text{st. } \angle - m(\text{st. } \angle - s);$$

and $u = m(\text{st. } \angle - s)$. In the same way $v = n(\text{st. } \angle - s)$.

Therefore $\triangle ABC : \triangle DEF = m : n = u : v$.

The demonstration is easily extended to the case of the incommensurability of the triangles ABC, DEF.

We may demonstrate in the same way that spherical triangles are as their spherical excesses.

If two of the angles of the spherical $\triangle$ are right angles, the third z will be the *excess* in question. Now, designating by p a great circle, this $\triangle$ is evidently

$$= \frac{z}{2\pi} \cdot \frac{p^2}{2\pi} \ (\S 32, \text{ VI}).$$

Consequently, any triangle of which the excess is z, is

$$= \frac{zp^2}{4\pi^2}.$$

43. Thus, in the system S, the area of a rectilineal $\triangle$ is expressed by means of the sum of its angles.

If AB increases to infinity, then (§42) the relation $\triangle ABC : \text{rt.} \angle - u - v$ will be constant. Now, $\triangle ABC \doteq BACN$ (§32, V) and $\text{rt.} \angle - u - v \doteq z$ (§1).

Therefore, $BACN : z =$

$\triangle ABC : (\text{rt.} \angle - u - v) = BAC'N' : z'$.

Moreover, we evidently have (§30)

$BDCN : BD'C'N' = r : r' = \tan z : \tan z'$.

Now, for $y' \doteq 0$, we have

$\frac{BD'C'N'}{BA'C'N'} \doteq 1$, and also $\frac{\tan z'}{z'} \doteq 1$.

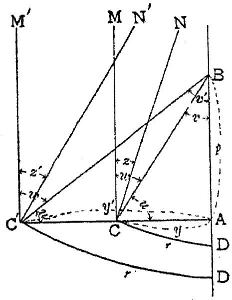


FIG. 39.

Consequently,

$$BDCN : BACN = \tan z : z.$$

But we have found (§32)

$$BDCN = r.i = i^2 \tan z.$$

Therefore,

$$BACN = z.i^2.$$

Designating henceforth, for brevity, every triangle of which the supplement of the angle-sum is z by $\triangle$, we will thus have

$$\triangle = z.i^2.$$

Hence we readily conclude that, if $OR \| AM$ and $RO \| AB$, the *area* contained between the straights OR, ST, BC (which is evidently the absolute limit of the area of rectilineal triangles increasing indefinitely, or the limit of $\triangle$ for $z \doteq \text{st.} \angle$), will be equal to $\pi i^2 = \odot i$ (in F).

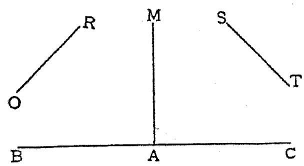


FIG. 40.

Designating this limit by $\square$, we will also have (§30) $\pi r^2 = \tan^2 z . \square = \odot r$ (in F) (§21) $= \odot s$ (§32, VI), representing the chord CD by s.

If now, by means of a perpendicular erected at the mid point of the given radius s of the circle in a plane (or of the radius of form L of the circle in F), we construct (§34) $DB \| CN$ and $\angle CDB = \angle DCN$; by dropping $CA \perp DB$, and erecting $CM \perp CA$, we shall get z; whence (§37), taking arbitrarily a radius of form L for unity, *we shall be able to determine geometrically* $\tan^2 z$, *by means of two uniform lines of the same curvature* (which, their extremities alone being given and their axes constructed, may evidently be treated as straights in seeking their common measure, and are in this respect the equivalent of straights).

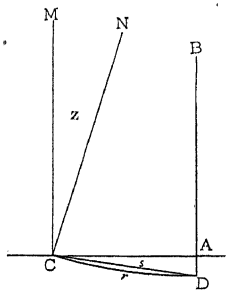


FIG. 41.

We can, moreover, construct as follows a quadrilateral, for example a regular quadrilateral, of area=□.

Take $ABC = $ rt. ∠, $BAC = \frac{1}{2}$ rt. ∠, $ACB = \frac{1}{4}$ rt. ∠, and $BC = x$.

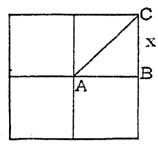


FIG. 42.

We can express X (§31, II) by simple square roots, and construct it (§37).

Knowing X, we can determine x (§38 or also §§29 and 35). The octuple of △ABC is evidently =□, and thus *a plane circle is geometrically squared by means of a rectilinear figure and of uniform lines of the same species* (that is to say of lines equivalent to straights as to their comparison to each other).

A circle of the surface F is planified in the same manner.

Thus either the Axiom XI *of Euclid is true or we have the geometric quadrature of the circle*, though nothing thus far decides which of the two propositions is real.

Whenever $\tan^2 z$ is either a whole number, or a rational fraction, of which the denominator (after reduction to the simplest form) is either a prime number of the form $2^m + 1$ (of which $2 = 2^0 + 1$ is a particular case), or a product of prime numbers of this form, of which each (with the exception of 2, which alone may enter any number of times) enters *only once* as factor, we can, by the theory of polygons given by Gauss (and for such values alone), construct a rectilineal figure $=\tan^2 z = \odot s$. Because the division of $\square$ (the theorem of §42 extending easily to any polygons) requires evidently the partition of a st. $\angle$, which (as we can demonstrate) is possible geometrically only under the preceding condition.

In all such cases, what precedes conducts easily to the desired end; and every rectilineal figure can be transformed geometrically into a regular polygon of n sides, if n is of the form indicated by Gauss.

It still remains, for the entire completion of our researches, to demonstrate the impossibility of deciding (without having recourse to some hypothesis) whether it is the system Σ, or some one of the systems S (and which one) which really exists. This we reserve for a more favorable occasion.

APPENDIX I.

REMARKS ON THE PRECEDING MEMOIR, BY WOLFGANG BOLYAI.

[From Vol. II of Tentamen, p. 380, et seq.]

I may be permitted to add here certain remarks appertaining to the author of the *Appendix*, who may pardon me if I do not always well express his thought.

The formulas of spherical trigonometry (demonstrated in the preceding memoir independently of Euclid's Axiom XI) *coincide with the formulas of plane trigonometry, when we consider* (to use a provisional method of speaking) *the sides of a spherical triangle as reals, those of a rectilineal triangle as imaginaries;* so that, when it is a question of trigonometric formulas, we may regard the plane as an imaginary sphere, taking for real sphere that in which sin rt. $\angle = 1$.

We demonstrate (§ 30) that there is a certain quantity i (in case of the non-existence of Euclid's axiom), such that the corresponding quantity I is equal to the base e of natural logarithms. In this case, we establish also (§31) the formulas of plane trigonometry, and in such manner (§32, VII) that the formulas are still true for the case of the reality of the axiom in question, taking the limit of the values for $i \doteq \infty$. Thus the Euclidean system is in a certain way the limit of the anti-Euclidean system for $i \doteq \infty$.

Take, in case of the existence of i, the unit$=i$, and extend the definitions of sine and of cosine to imaginary arcs, so that, p designating an arc whether real or imaginary, the expression

$$\frac{e^{p\sqrt{-1}}+e^{-p\sqrt{-1}}}{2}$$ is to be always called

the *cosine* of p, and the expression

$$\frac{e^{p\sqrt{-1}}-e^{-p\sqrt{-1}}}{2\sqrt{-1}}$$ the *sine* of p.

We shall have for q real

$$\frac{e^{q}-e^{-q}}{2\sqrt{-1}}=\frac{e^{-q\sqrt{-1}.\sqrt{-1}}-e^{q\sqrt{-1}.\sqrt{-1}}}{2\sqrt{-1}}=\sin(-q\sqrt{-1})$$

$$=-\sin(q\sqrt{-1}),$$ and in like manner

$$\frac{e^{q}+e^{-q}}{2}=\frac{e^{-q\sqrt{-1}.\sqrt{-1}}+e^{q\sqrt{-1}.\sqrt{-1}}}{2}=\cos(-q\sqrt{-1})=\cos(q\sqrt{-1})$$

admitting that, in the imaginary circle as in the real circle, the sines of two arcs equal but of contrary sign are equal and of opposite sign, and that the cosines of two arcs equal but of opposite sign are equal and of the same sign.

We demonstrate, in §25, absolutely, that is to say independently of the axiom in question, that, in every rectilineal triangle *the sines of the angles are to each other as the circles which have for radii the sides opposite to these angles.*

We demonstrate, besides, for the case of the existence of the quantity i, that the circle of radius y is equal to $\pi i\left(e^{\frac{y}{i}}-e^{-\frac{y}{i}}\right)$, which, for $i=1$, becomes

$$\pi(e^{y}-e^{-y}).$$

Consequently (§31), in a right-angled rectilineal triangle of which the sides of the right angle are a and b, and the hypothenuse c, and of which the angles opposite to the sides a, b, c are α, β, rt. $\angle$, we have (for $i=1$).

From I,

$$1:\sin \alpha = \pi(e^{c}-e^{-c}):\pi(e^{a}-e^{-a});$$

and consequently

$$1:\sin \alpha = \frac{e^{c}-e^{-c}}{2\sqrt{-1}}:\frac{e^{a}-e^{-a}}{2\sqrt{-1}}$$

$$=-\sin(c\sqrt{-1}):-\sin(a\sqrt{-1})=\sin(c\sqrt{-1}):\sin(a\sqrt{-1});$$

From II,

$$\cos \alpha:\sin \beta = \cos(a\sqrt{-1}):1;$$

From III,

$$\cos(c\sqrt{-1})=\cos(a\sqrt{-1}).\cos(b\sqrt{-1}).$$

These formulas, as also all the formulas of plane trigonometry deducible from them, coincide completely with the formulas of spherical trigonometry, to this extent that if, for example, the sides and the angles of a right-angled rectilineal triangle are designated by the same letters as those of a right-angled spherical triangle, the sides of the rectilineal triangle are to be divided by $\sqrt{-1}$ to obtain the formulas relative to the spherical triangle.

So we get, for a spherical triangle,

by I, $\quad 1:\sin \alpha = \sin c:\sin a;$

by II, $\quad 1:\cos a = \sin \beta:\cos \alpha;$

by III, $\quad \cos c = \cos a \cos b.$

As the reader may be stopped by the omission of a demonstration (in §32 at end) it will not be useless to show, for example, how from the equation

$$e^{\frac{c}{i}}+e^{-\frac{c}{i}}=\frac{1}{2}\left(e^{\frac{a}{i}}+e^{-\frac{a}{i}}\right)\left(e^{\frac{b}{i}}+e^{-\frac{b}{i}}\right)$$

we deduce the formula

$$c^2=a^2+b^2,$$

or the theorem of Pythagoras for the Euclidean system.

It is probably thus that the author arrived at it, and the other consequences follow in a similar manner.

We have, by the known formula,

$$e^{\frac{k}{i}}=1+\frac{k}{i}+\frac{k^2}{2i^2}+\frac{k^3}{2.3.i^3}+\frac{k^4}{2.3.4.i^4}+\;\ldots\;,$$

$$e^{-\frac{k}{i}}=1-\frac{k}{i}+\frac{k^2}{2i^2}-\frac{k^3}{2.3.i^3}+\frac{k^4}{2.3.4.i^4}-\;\ldots\;,$$ and consequently

$$e^{\frac{k}{i}}+e^{-\frac{k}{i}}=2+\frac{k^2}{i^2}+\frac{k^4}{3.4.i^4}+\;\ldots\;=2+\frac{k^2+u}{i^2},$$

designating by $\frac{u}{i^2}$ the sum of all the terms which follow $\frac{k^2}{i^2}$; and we have $u \doteq 0$, when $i \doteq \infty$. For all the terms which follow $\frac{k^2}{i^2}$, on being divided by i^2, (that is the factor i^2 being taken out of the denominator), will have for first term $\frac{k^4}{3.4i^2}$; and as the ratio of a term to the preceding is throughout $<\frac{k^2}{i^2}$, the sum is less than it would be, if this ratio were $=\frac{k^2}{i^2}$, that is to say less than

$$\frac{k^4}{3.4.i^2}:\left(1-\frac{k^2}{i^2}\right)=\frac{k^4}{3.4.(i^2-k^2)},$$

a quantity which evidently $\doteq 0$ when $i \doteq \infty$.

From the equation

$$e^{\frac{c}{i}}+e^{-\frac{c}{i}}=\tfrac{1}{2}\left(e^{\frac{a+b}{i}}+e^{-\frac{a+b}{i}}+e^{\frac{a-b}{i}}+e^{-\frac{a-b}{i}}\right)$$

there results (calling w, v, λ quantities analogous to u)

$$2+\frac{c^2+w}{i^2}=\tfrac{1}{2}\left(2+\frac{(a+b)^2+v}{i^2}+2+\frac{(a+b)^2+\lambda}{i^2}\right),$$ whence

$$c^2=\frac{a^2+2ab+b^2+a^2-2ab+b^2+v+\lambda-w}{2},$$ a quantity

which $\doteq a^2+b^2$.

APPENDIX II.

SOME POINTS IN JOHN BOLYAI'S APPENDIX COMPARED WITH LOBATSCHEWSKY, BY WOLFGANG BOLYAI.

[From *Kurzer Grundriss*, p. 82.]

Lobatschewsky and the author of the *Appendix* each consider two points A, B, of the sphere-limit, and the corresponding axes ray AM, ray BN (§23).

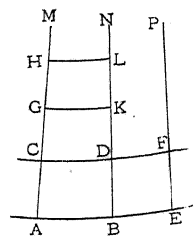


FIG. 43.

They demonstrate that, if α, β, γ designate the arcs of the circle limit AB, CD, HL, separated by segments of the axis AC=1, AH=x, we have

$$\frac{\alpha}{\gamma}=\left(\frac{\alpha}{\beta}\right)^{x}.$$

Lobatschewsky represents the value of $\frac{\gamma}{\alpha}$ by e^{-x}, e having some value >1, dependent on the unit for length that we have chosen, and able to be supposed equal to the Naperian base.

The author of the *Appendix* is led directly to introduce the base of natural logarithms.

If we put $\frac{\alpha}{\beta}=\delta$, and γ, γ' are arcs situated at the distances y, i from α, we shall have

$$\frac{\alpha}{\gamma}=\delta^{y}=Y,\ \frac{\alpha}{\gamma'}=\delta^{i}=I, \text{ whence } Y=I^{\frac{y}{i}}.$$

He demonstrates afterward (§29) that, if u is the angle which a straight line makes with the perpendicular to its parallel, we have

$$\mathrm{Y}=\cot \tfrac{1}{2} u.$$

Therefore, if we put $z=\frac{\pi}{2}-u$, we have

$$\mathrm{Y}=\tan(z+\tfrac{1}{2}u)=\frac{\tan z+\tan\frac{1}{2}u}{1-\tan z \tan\frac{1}{2}u},$$

whence we get, having regard to the value of $\tan \frac{1}{2}u=\mathrm{Y}^{-1}$,

$$\tan z=\tfrac{1}{2}(\mathrm{Y}-\mathrm{Y}^{-1})=\tfrac{1}{2}\left\{\mathrm{I}^{\frac{y}{i}}-\mathrm{I}^{-\frac{y}{i}}\right\} \quad (\S 30).$$

If now y is the semi-chord of the arc of circle-limit $2r$, we prove (§30) that $\frac{r}{\tan z}$=constant.

Representing this constant by i, and making y tend toward zero, we have

$$\frac{2r}{2y} \doteq 1, \text{ whence}$$

$$2y \doteq 2\, i \tan z \doteq i\,\frac{\mathrm{I}^{\frac{2y}{i}}-1}{\mathrm{I}^{\frac{y}{i}}}$$

or putting $\frac{2y}{i}=k$, $\mathrm{I}=e^{l}$,

$$kl^{\frac{y}{i}} \doteq e^{kl}-1 \doteq kl(1+\omega),$$

ω being infinitesimal at the same time as k, Therefore, for the limit, $1=l$ and consequently $\mathrm{I}=e$.

The circle traced on the sphere-limit with the arc r of the curve-limit for radius, has for length $2\pi r$. Therefore,

$$\bigcirc y=2\pi r=2\pi\, i \tan z=\pi i(\mathrm{Y}-\mathrm{Y}^{-1}).$$

In the rectilineal $\triangle$ where α, β designate the angles opposite the sides a, b, we have (§25)

$$\sin \alpha : \sin \beta=\bigcirc a : \bigcirc b=\pi i(\mathrm{A}-\mathrm{A}^{-1}) : \pi i(\mathrm{B}-\mathrm{B}^{-1})$$
$$=\sin(a\sqrt{-1}) : \sin(b\sqrt{-1}).$$

Thus in plane trigonometry as in spherical trigonometry, the sines of the angles are to each other as the sines of the opposite sides, only that on the sphere the sides are reals, and in the plane we must consider them as imaginaries, just as if the plane were an imaginary sphere.

We may arrive at this proposition without a preceding determination of the value of I.

If we designate the constant $\frac{r}{\tan z}$ by q, we shall have, as before

$$\bigcirc y = \pi q (\mathrm{Y} - \mathrm{Y}^{-1}),$$

whence we deduce the same proportion as above, taking for i the distance for which the ratio I is equal to e.

If axiom *XI* is not true, there exists a determinate i, which must be substituted in the formulas.

If, on the contrary, this axiom is true, we must make in the formulas $i = \infty$. Because, in this case, the quantity $\frac{a}{r} = \mathrm{Y}$ is always $= 1$, the sphere-limit being a plane, and the axes being parallel in Euclid's sense.

The exponent $\frac{y}{i}$ must therefore be zero, and consequently $i = \infty$.

It is easy to see that Bolyai's formulas of plane trigonometry are in accord with those of Lobatschewsky.

Take for example the formula of §37,

$$\tan \Pi(a) = \sin \mathrm{B} \tan \Pi(p),$$

a being the hypothenuse of a right-angled triangle, p one side of the right angle, and B the angle opposite to this side.

Bolyai's formula of §31, I, gives

$$1 : \sin \mathrm{B} = (\mathrm{A} - \mathrm{A}^{-1}) : (\mathrm{P} - \mathrm{P}^{-1}).$$

Now, putting for brevity, $\frac{1}{2}\Pi(k) = k'$, we have $\tan 2p' : \tan 2a'$ $= (\cot a' - \tan a') : (\cot p' - \tan p') = (\mathrm{A} - \mathrm{A}^{-1}) : (\mathrm{P} - \mathrm{P}^{-1})$ $= 1 : \sin \mathrm{B}.$

APPENDIX III.

LIGHT FROM NON-EUCLIDEAN SPACES ON THE TEACHING OF ELEMENTARY GEOMETRY.

By G. B. Halsted.

The preface to my Elements of Geometry, 1885, says "that around the word 'distance' centers the most abstruse advance in pure science and philosophy."

Recently R. A. Roberts, in his "Modern Mathematics," gives as one of the two main roots from which modern mathematical thought springs, the recognition of the fact that angles and distances in the Euclidean experiential geometry depend upon a certain absolute curve of the second order.

As foreshadowed by Bolyai and Riemann, founded by Cayley, extended and interpreted for hyperbolic, parabolic, elliptic spaces by Klein, and now recast and applied to mechanics by Sir Robert Ball, this projective metrics may in truth be looked upon as the very soul and characteristic of what is highest and most peculiarly modern in all the bewildering range of mathematical achievement.

It permeates like a vital essence, and for questions of method, of teaching, of exposition it is a final criterion. Nearly all mathematicians have already fallen into rank as holding that number is wholly a creation of the human intellect, while on the contrary geometry has an empirical element. Of a number of possible geometries we cannot say *a priori* which shall be that of our actual space, the space in which we move. Of course an advance so important, not only for mathematics but for philosophy, has had some metaphysical opponents, and as long ago as 1878 I mentioned in my Bibligraphy of Hyper-Space and Non-Euclidean Geometry (American Journal of Mathematics Vol. I, 1878, Vol. II, 1879) one of these, Schmitz--Dumont, as a sad paradoxer, and another, J. C. Becker, both of whom would ere this have shared the oblivion of still more antiquated fighters against the light, but that Dr. Schotten, praiseworthy for the very attempt at a comparative planimetry, happens to be himself a believer in the *a priori* founding of geometry, while his American reviewer, Mr. Ziwet, happens to confuse what would be good in a book written for the very necessary preparatory or propaedentric courses in intuitive geometry, with what would be good in a treatise professing to deduce Euclidean geometry from only the necessary assumptions.

He says, "we find that some of the best German text books do not try at all to define what is space, or what is a point, or even what is a straight line." Do any German geometries define space? I never remember to have met one.

In experience, what comes first is a bounded surface, with its boundaries, lines, and their boundaries, points. Are the points whose definitions are omitted anything different or better?

Dr. Schotten regards the two ideas "direction" and "distance" as intuitively given in the mind and as so simple as to not require definition.

As to what Webster's Dictionary says of the meaning of the English word "direction", Professor Cajori has honored me by a quotation on page 383 of his admirable History of Mathematics in the United States, and only today I saw mention of an accident caused while two jockeys were speeding around a track in opposite directions, and also chanced on page 87 of Richardson's Euclid, 1891, to read "The sides of the figure must be produced in the same direction of rotation; . . . going round the figure always in the same direction."

No wonder that when Mr. Ziwet had written: "he therefore bases the definition of the straight line on these two ideas," he stops, modifies, and rubs that out as follows, "or rather recommends to elucidate the intuitive idea of the straight line possessed by any well-balanced mind by means of the still simpler ideas of direction" [in a circle] "and distance" [on a curve]. If this is meant for an introductory geometry-for-beginners, all well and good. Elucidate any intuition possessed by the well-balanced baby-mind by anything still simpler which you may happen to think will elucidate it.

But when we come to geometry as a science, as foundation for work like that of Cayley and Ball, I think with Professor Chrystal: "It is essential to be careful with our definition of a *straight line*, for it will be found that virtually the properties of the straight line determine the nature of space.

Our definition shall be that two points *in general* determine a straight line, or that in general a straight line cannot be made to pass through *three* given points."

We presume that Mr. Ziwet glories in that unfortunate expression "a straight line is the shortest distance between two points," still occurring in Wentworth (New Plane Geometry, page 33,) even after he has said, page 5, "the length of the straight line is called the *distance* between two points." If the *length* of the one straight line between two points is the distance between those points how can the straight line itself be the *shortest* distance. If there is only one distance, it is the longest as much as the shortest distance, and if it is the *length* of this shorto-longest distance which is the distance then it is not the straight line itself which is the longo-shortest distance.

But Wentworth also says "Of all lines joining two points the *shortest* is the straight line."

This general comparison involves the measurement of curves, which involves the theory of limits, to say nothing of ratio. The very ascription of length to a curve involves the idea of a limit. And then to introduce this general axiom only to prove a very special case of itself, that two sides of a triangle are together greater than the third, is surely bad logic, bad pedagogy, bad mathematics.

This latter theorem, according to the first of Pascal's rules for demonstrations, should not be proved at all, since every dog knows it. Well and good in our geometry-for-beginners, to which alone Pascal's rules apply; but to this objection, as old as the sophists, Simson long ago answered for the science of geometry, that the number of assumptions ought not to be increased without necessity, or as Dedekind has it: "*Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.*"

But Mr. Ziwet could correct one of his misapprehensions by looking into Wentworth's book, namely the mistaken idea that American "text books begin with several pages of definitions *to be committed to memory*, followed by a page of axioms *again to be committed to memory*." Wentworth carefully reproduces, whenever he uses them, preceding definitions, axioms, theorems.

It is worth notice that the mistake made in our Century Dictionary, the confusion of hyperbolic with elliptic geometry, is made also on page 186 of Rebiere's enjoyable "Mathématiques et Mathématiciens," 1889, where he says: "De là des *geometries non-euclidiennes* ou la somme des angles d'un triangle n'est plus égale à deux droits: dans celle de Rieman, elle est plus petite que deux droit et dans celle de Lobatschewski, elle est plus grande." Note also that, Frenchman-like, both the proper names are here mis-spelled. May we not fear that here also is a teacher of mathematics who never has read Lobatschewsky's immortal Essay on Parallels? Contrast a distinguished Englishman, Professor Levett, who says: "It is many years since I first made acquaintance with this great work, and I am delighted to see that the good cause of sound geometrical learning has been advanced by the appearance of an English translation. I believe that I am one of the very few schoolmasters who have read the essay with pupils. King Edward's school boys are brought up in the true faith as to the sum of the angles of a triangle."

The brilliant American, Professor W. B. Smith, (Ph. D., Goettingen) has just written: "Nothing could be more unfortunate than the attempt to lay the notion of Direction at the bottom of Geometry."

Was it not this notion which led so good a mathematician as John Casey to give as a demonstration of a triangle's angle-sum the procedure called "a *practical* demonstration" on page 87 of Richardson's Euclid, and there described as "laying a 'straight edge' along one of the sides of the figure, and then turning it round so as to coincide with each side in turn."

This assumes that a straight line may be translated without rotation, which assumption readily comes to view when you try the procedure in two-dimensional double-elliptic geometry, our familiar two-dimensional spherics. It is of the greatest importance for every teacher to know and connect the commonest forms of assumption equivalent to Euclid's Axiom XI. If in a plane two straight lines perpendicular to a third can never meet, are there others, not both perpendicular to any third, which can never meet? Euclid's Axiom XI is the assumption *No*. Playfair's answers *no* more simply, But the very same answer is given by the common assumption of our geometries, usually unnoticed, that a circle may be passed through any three non-collinear points.

This equivalence was first shown by Bolyai, who looks upon this as the simplest form of the assumption. Lobatschewsky's form is, the existence of any finite triangle whose angle-sum is a straight angle; or the existence of a plane rectangle; or that, in triangles, the angle-sum is constant.

One of Legendre's forms was that through every point within an angle a straight line may be drawn which cuts both arms.

But Legendre never saw through this matter because he had not, as we have, the eyes of Bolyai and Lobatschewsky to see with. The same lack of their eyes has caused the author of the charming book "Euclid and His Modern Rivals," to give us one more equivalent form: "In any circle, the inscribed equilateral tetragon is greater than any one of the segments which lie outside it," (A New Theory of Parallels by C. L. Dodgson, 3d. Ed., 1890).

NOTE ON THE TRANSITION CURVE.

By Prof. W. H. Echols, University of Virginia.

I.

In *Scientiæ Baccalaureus*, Vol. I., No. 1, in the article *The Railway Transition Curve*, TABLE I, is altogether wrong. In correcting this error, I take advantage of the opportunity to present anew the reduction of the formulae there given, to present a further development of the system of Transition Curves and to call the attention of any one interested in this subject to a most interesting and highly valuable article in No. 5; *The Technograph* of the University of Illinois by Professor Talbot.

Prof. Talbot has developed the Transition Spiral, as defined p. 17, No. 1, in this journal, while I have developed the "Transition Curve", using this name to mean the "Taper Curve" as defined by Prof. Talbot.

In my article of March, 1890, referred to above, I defined the Transition Curve as follows: "A Transition Curve is one whose curvature increases per unit of arc in arithmetical progression, or whose change of curvature per unit arc is constant."

Using the same notation as there used, let R and D be the radius and degree of the main curve, united to the tangent by a series of arcs of $r_1, \ldots r_n$ and central angles $d_1, \ldots d_n$ respectively. Then we have for the tangent distance of the curve,

$$T=(R+p)\tan\tfrac{1}{2}I+q.$$

When p and q are the distances of the point of contact of the main curve with a tangent parallel to the initial tangent, measured from the initial tangent and from the P. C. along the initial tangent respectively.

From the figure in No. 1, we have,

$$q=\left\{\begin{array}{l} (r_1-r_2)\sin d_1, \\ +(r_2-r_3)\sin(d_1+d_2) \\ + \;\ldots \\ +(r_n-R)\sin(d_1+\;\ldots\; d_n). \end{array}\right.$$

Putting $\qquad r_1=2r_2=3r_3=\;\ldots\;=nr_n,$

and $\qquad d_1=\tfrac{1}{2}d_2=\tfrac{1}{3}d_3=\;\ldots\;=\tfrac{1}{n}d_n.$

We have,

$$q=\tfrac{1}{2}r_1\left\{\begin{array}{l} \sin d_1 \\ +\tfrac{1}{3}\sin 3d_1 \\ +\tfrac{1}{6}\sin 6d_1 \\ +\;\ldots \\ +\dfrac{1}{\tfrac{1}{2}n(n-1)}\sin\tfrac{1}{2}(n-1)nd_1 \\ +2\left(\dfrac{1}{n}-\dfrac{R}{r_1}\right)\sin\tfrac{1}{2}n(n+1)d_1. \end{array}\right.$$

If we put in the circular measures for the sines of these small angles, we get,

$$q=\tfrac{1}{2}r_1d_1\frac{\pi}{180}\left\{n-1+2\left(\frac{1}{n}-\frac{R}{r_1}\right)\frac{n(n+1)}{2}\right\},$$

$$=nr_1d_1\frac{\pi}{180}\left\{1-(n+1)\frac{R}{2r_1}\right\},$$

$$=nr_nd_n\frac{\pi}{180}\left\{1-(n+1)\frac{R}{2nr_n}\right\}.$$

In particular if $R = r_n$,

$$q = \frac{\pi}{180}\,\frac{n-1}{2}\,r_n d_n.$$

If $r_1 = \frac{180}{\pi} \times 100 = 5730$, and $d_1 = \frac{1}{2}^\circ$.

Then, in general,

$$q = 50n\left(1 - \frac{n+1}{2D}\right);$$

and if D be integral $= n+1$,

$$q = 25(D-1) = 25n.$$

In like manner, we have,

$$p = \begin{cases} r_1(1 - \cos d_1) - r_2(1 - \cos d_1) \\ + r_2(1 - \cos \overline{d_1 + d_2}) - r_3(1 - \cos \overline{d_1 - d_2}) \\ + \;.\;.\;.\;. \\ + r_n[1 - \cos(d_1 + \;.\;.\;.\; d_n)] - R[1 - \cos(d_1 + \;.\;.\;.\; d_n)], \end{cases}$$

$$= r_1 \begin{cases} (r_1 - r_2)\, 2 \sin^2 \tfrac{1}{2} d_1 \\ + (r_2 - r_3)\, 2 \sin^2 \tfrac{1}{2}(d_1 + d_2) \\ + \;.\;.\;.\;. \\ + (r_{n-1} - r_n)\, 2 \sin^2 \tfrac{1}{2} \dfrac{n(n-1)}{2} d_1 \\ + (r_n - R)\, 2 \sin^2 \tfrac{1}{2} \dfrac{n(n+1)}{2} d_1. \end{cases}$$

Putting as before, $r_1 = n r_n$, and $d_1 = \frac{1}{n} d_n$, etc., we have,

$$p = \begin{cases} \sin^2 \tfrac{1}{2} d_1 \\ + \tfrac{1}{3} \sin^2 \tfrac{1}{2} 3 d_1 \\ + \;.\;.\;.\;. \\ + \dfrac{1}{\tfrac{1}{2} n(n-1)} \sin^2 \tfrac{1}{2} \dfrac{n(n-1)}{2} d_1. \\ + 2\left(\dfrac{1}{n} - \dfrac{R}{r_1}\right) \sin^2 \tfrac{1}{2} \dfrac{n(n+1)}{2} d_1. \end{cases}$$

$$= \left(\frac{\pi}{180}\right)^2 \frac{r_1 d_1^2}{4}\left[1 + 3 + 6 + \;.\;.\;. + \tfrac{1}{2} n(n-1) + \left(\frac{1}{n} - \frac{R}{r_1}\right) \frac{n^2(n+1)^2}{2}\right],$$

$$= \left(\frac{\pi}{180}\right)^2 \frac{r_1 d_1^2}{4}\left\{\frac{n(n+1)(n+2)}{6} - \frac{n(n+1)}{2} + \left(\frac{1}{n} - \frac{R}{r_1}\right) \frac{n^2(n+1)^2}{2}\right\},$$

$$= \left(\frac{\pi}{180}\right)^2 \frac{r_1 d_1^2 n(n+1)}{4 \cdot 2}\left\{\frac{n-1}{3} + n(n+1)\left(\frac{1}{n} - \frac{R}{r_1}\right)\right\}.$$

In particular, if $R=r_n$, we have,

$$p=n(n^2-1)\left(\frac{\pi}{180}\right)^2\frac{r_1d_1^2}{24}.$$

If, however, D is not integral and $n+1$, but is n degrees and m minutes, we have, for $r_1=5730$ and $d_1=\frac{1}{2}$,

$$p=\frac{n^3}{54}+\frac{mn^2}{1000}.$$

The Table I, p. 14, Vol. I, No. 1, should be computed from this formula, since the table there given *is wholly wrong*; it is given here correctly.

Table I.

D	1°	2°	3°	4°	5°	6°	7°
00′	0	0,1	0,4	1,1	2,2	4,0	6,2
10′	0	0,1	0,5	1,3	2,5	4,3	6,7
20′	0	0,2	0,7	1,4	2,8	4,7	7,1
30′	0,1	0,2	0,8	1,6	3,1	5,1	7,8
40′	0,1	0,3	0,9	1,8	3,4	5,5	8,3
50′	0,1	0,4	0,9	2,0	3,7	5,9	8,8
Diff	0,0	0,0	0,1	0,2	0,3	0,4	0,5

The formula from which this table is computed is so simple that no table is necessary.

In like manner the deflections from the initial tangent to set the n points on the transition, tabulated in Table II, p. 15, No. 1, may be expressed in a simple formula as follows, in minutes,

$$V_1=\tfrac{5}{2}[2n^2+3n+1].$$

In same way,

$$D_c=\tfrac{5}{2}+10n.$$
$$D_t=\tfrac{5}{2}[(2n)^2+3n-1].$$

Again, in running in the finial transition curve from the end of the main curve, we find the necessary deflections tabulated on the same page referred to above, in the TABLE III. Noticing that these series of angles may be summed and expressed in a formula by making use of the principles of "*finite differences*", we say that the deflection in minutes from the terminal tangent of the main curve to set the mth point on an n chord transition curve, is,

$$V_t=\frac{10}{4}[m(6n+3-2m)-1].$$

This does away with TABLE III.

To get working formulae for the coördinates of a point on the transition curve (see x and y, TABLE II) we notice that the mth chord of the transition makes with initial tangent the angle $\tfrac{1}{2}m^2d_1$ or $\tfrac{1}{4}m^2$ degrees, we have for the projections of the chords on the initial tangent and a normal to it

$$\Delta x=50\cos\left(\frac{m}{2}\right)^2;\qquad \Delta y=50\sin\left(\frac{m}{2}\right)^2$$

$$\therefore\ \Delta x=50\left[1-\tfrac{1}{2}\left(\frac{\pi}{180}\right)^2\left(\frac{m}{2}\right)^4\right],\text{ nearly.}$$

$$\therefore\ x=\sum_0^m(\Delta x),$$

$$=50\,m-\left(\frac{\pi}{180}\right)^2\frac{50}{32}\sum_0^m m^4,$$

$$\sum_0^m m^4=\frac{1}{30}(6n^5+15n^4),\text{ using the two highest powers.}$$

$$\therefore\ x=50m-\frac{10}{32}\left(\frac{\pi}{180}\right)^2(m^5+\tfrac{5}{2}m^4),$$

$$=50m-\frac{1}{10^4}(m^5+\tfrac{5}{2}m^4),\text{ nearly enough.}$$

Or the distance along the tangent is $\frac{1}{10^4}(m^5+\frac{5}{3}m^4)$ less than the distance along the curves.

Likewise for y,

$$\Delta y=50\frac{\pi}{180}\left(\frac{m}{2}\right)^2,$$

$$y=\sum_0^m(\Delta y)=\frac{50\pi}{180}\frac{1}{4}\sum_0^m m^2$$

$$\sum_0^m m^2=\tfrac{1}{6}(2m^3+3m^2), \text{ using highest powers.}$$

$$\therefore\ y=\frac{5\pi}{216}(m^3+\tfrac{3}{2}m^2),$$

$$=0{,}073(m^3+\tfrac{3}{2}m^2).$$

To recapitulate:

In order to unite two tangents meeting at I° by a D° curve and to pass from the tangents to the curve by a series of arcs of 50 feet whose curvatures are respectively those of 1°, 2°, 3° . . . n° curves, n being the number of integral degrees in D° and m the number of minutes left over.

We have

(1) $$T=(R+p)\tan\tfrac{1}{2}I+q$$

Where $$p=\frac{1}{54}n^3+0.001\ m\,n^2.$$

$$q=50\,n\left(1-\frac{n+1}{2D}\right).$$

[If D is of integral degree then $m=0$, $D=n+1$ and

$$T=(R+\frac{1}{54}n^3)\tan\tfrac{1}{2}I+25\,n].$$

The vernier angles for setting the n points on the transition from its initial point are in minutes

(2) $$V_i = \tfrac{5}{2}(2n^2 + 3n + 1),$$

and from its terminal point

(3) $$V_t = \tfrac{10}{4}[m(6n + 3 - 2n) - 1].$$

A tangent to the spiral at the nth point makes with the initial tangent the angle

(4) $$15n(n + 1).$$

The distance from the beginning of curve, measured along the tangent, to the mth point is less than the length of the curve to that point by

(5) $$\frac{m^5 + \frac{3}{2}m^4}{104} \text{ or nearly } \frac{m^5}{104},$$

and the offset from the tangent is

(6) $$0{,}073(m^3 + \tfrac{3}{2}m^2) \text{ or nearly } \frac{3}{4}\,\frac{m^3}{10}.$$

The transition moves the mid point of curve

$$p \sec \tfrac{1}{2}\,\mathrm{I} \text{ or } \frac{1}{54}\, n^3 \sec \tfrac{1}{2}\,\mathrm{I}^{\circ}$$

further from the intersection of the tangents.

Thus no tables are needed to run in such a curve (other than table of tanget for getting T). It will in general permit easier running if the small tables such as I, II and III are used.

However n being an integer less than 8 the formulae are easy.

II.

Prof. Talbot has so nicely developed the Transition Spiral for a railway curve that I am tempted to push the taper curve to its limit *without the calculus* for sake of the interest that may be had in the deduction of the formulae as well as for such value as it may have for practical purposes.

Let the design be to unite a pair of "tangents" by a circular curve in such a manner that the transition from tangent to curve is made by another curve whose curvature *per foot of* length increases uniformly from tangent to main curve. The curvature of main curve and transition being the same at their point of contact.

Then the central angle of the last *foot* of the transition is $\frac{1}{100}$ D (letting D be the degree of the main curve in *minutes*), and S the length in feet of the transition. If δ is the number of minutes in the central angle of the first foot and i the constant increment of central angle per foot, we have

$$\delta = i = \frac{D}{100\,S}.$$

And as before

$$T = (R+p)\tan \tfrac{1}{2} I + q.$$

Where p and q are as before,

$$p = \frac{n(n^2-1)}{24}\left(\frac{\pi}{180}\right)^2 r_n d_n d_1,$$

becomes

$$p = 0{,}0000001212\ S^2 D,$$
$$= 0{,}0727\ L^2 D_c.$$

Where L is length of transition in chains (100′) and D_c degree of main curve in degrees.

Also

$$q = \frac{\pi}{180}\,\frac{n-1}{2}\,r_n d_n,$$

becomes

$$q = \tfrac{1}{2}(S-1).$$

Since $\qquad n = S$, and $r_n d_n = \frac{1}{100}\,RD = \frac{180}{\pi}$.

The sth central angle is $s\delta$ and the sth foot of the transition makes with the tangent at its end the angle $\frac{1}{2}s\delta$. The angle which the tangent at s makes with the initial tangent is

$\frac{1}{2}\partial s(s+1)$, therefore the sth chord makes $\frac{1}{2}\partial s^2$ minutes with the initial tangent. The projections on this tangent and a normal to it are

$$\Delta x = \cos \tfrac{1}{2}\partial s^2; \quad \Delta y = \sin \tfrac{1}{2}\partial s^2.$$

$$\therefore \quad x = \sum_0^s{}' (\Delta x)$$

$$= s - \frac{\partial^2}{40(3428)^2} \sum_0^s{}' s^4$$

$$= s - \frac{2115}{10^{12}} \partial^2 (s^5 + \tfrac{5}{3}s^4),$$

$$= s - 0{,}00076\, l^5 \frac{D_c^{\,2}}{L^2}, \text{ nearly.}$$

If l be s in chains.

In like manner for y

$$y = \sum_0^s{}' (\Delta y),$$

$$= \tfrac{1}{2} \frac{\partial}{3438} \sum_0^s{}' s^2,$$

$$= \frac{\partial}{6876} (\tfrac{1}{3}s^3 + \tfrac{1}{2}s^2),$$

$$= 0{,}00004848\partial (s^3 + \tfrac{3}{2}s^2)$$

$$= 0{,}291\, l^3 \frac{D_c}{L}.$$

If here $s = S$, the offset Y at the end of the curve is

$$Y = 0{,}0000004848\, DS^2.$$

But $$p = 0{,}0000001212\, DS^2.$$

Therefore, $$Y = 4p.$$

The long chord of the spiral makes with the tangent the angle θ, such that

$$\tan \theta = y/x.$$

Since θ is small, we have

$$\theta = 3438\,\frac{y}{x},$$
$$= 0{,}16666\,\delta s(s+\tfrac{3}{2}) \text{ nearly},$$
$$= \tfrac{1}{3}\,\tfrac{1}{2}\,\delta s(s+\tfrac{3}{2}).$$

But the tangent at s makes $\varphi = \frac{1}{2}\delta s(s+1)$ with the initial tangent, hence

$$\theta = \tfrac{1}{3}\varphi + \tfrac{1}{4}\delta s.$$

It would be useless to carry the development of this curve any further since Professor Talbot has worked the system up so completely.

It may be suggested, however, that this system be applied to that particular spiral whose length in feet is half the number of minutes in the degree of the main curve. This gives a transition of about the proper length, such as is employed in practice. In such a curve the change of deflection is one minute per foot, giving very convenient and satisfactory dimensions. The (approximate) formulae for such a transition, good up to an 8° curve, are

$$S = \tfrac{1}{2}\,D,$$

$$\delta = i = 0{,}02,$$

$$p = \frac{1}{10^6}\,\frac{S^3}{4} = \tfrac{1}{4}L^3,$$

$$q = \tfrac{1}{2}(S-1),$$

$$x = S - \frac{846}{10^{15}}\,s^5 = s - \frac{8l^3}{1000},$$

$$y = \frac{97}{10^8}\,s^3 = l^3 - \frac{3}{100}\,l^3,$$

$$\theta = \tfrac{1}{3}\varphi + \tfrac{1}{2}l = \frac{s^2}{300} + \tfrac{1}{2}l.$$

$$\varphi = \frac{s^2+s}{100} = l(s+1).$$

The angle Ψ for deflecting from long chord to the tangent at l on the spiral is

$$\Psi=\varphi-\theta=2s-\tfrac{3}{2}l.$$

For working purposes these are close enough,

$$p=\tfrac{1}{4}L^3;\ x=s-(\tfrac{2}{10}l)^3;\ y=l^3;\ s=\tfrac{1}{300}s^2;\ \varphi=0{,}01(s^2+s).$$

The tangent may be drawn

$$T=\left\{R+7\left(\frac{D_\circ}{10}\right)^3\right\}\tan\tfrac{1}{2}I+15\,D_\circ-0.5.$$

Where evidently the $D_\circ$ in the brace need only be used to the nearest integral degree.

There remains nothing to be considered unless it is the vernier angle or deflection from the terminal tangent of the main curve to set in a point on the terminal transition.

Let accents l', θ', s', etc., indicate tan the same quantities running from this end of the curve as is indicated by the unaccented letters when running from the other end.

The capitals Θ, Φ, L, S, etc., refer to the whole transition. Then in the triangle formed by the two ends of the curve and a point on it, we have, since the angles involved are small and the arcs s and s' nearly enough equal to their chords,

$$\frac{\theta'-s'}{\Theta-s}=\frac{s}{s'}=\frac{L-l'}{l'}.$$

In this equation, we have,

$$\theta'=\Phi-\Theta=2\Theta-\tfrac{3}{2}L,$$

$$\Theta=\tfrac{100}{3}L^2;\ \theta=\tfrac{100}{3}(L-l')^2.$$

Substituting these values and reducing, we get, in minutes

$$\theta'=100\,l'L-\tfrac{100}{3}l'^2\ \tfrac{3}{2}L.$$

If $l'=L$ we get

$$\theta=\tfrac{2}{3}100\,L^2-\tfrac{3}{2}L$$
$$=2\,\theta-\tfrac{3}{2}L,$$

as should be.

The error of the first assumption manifests itself in the term $\frac{3}{2}L$, a constant, for $l'=0$ gives $\theta'=\frac{3}{2}L$ which is wrong. If, in order to get correct results at the limits, we put $\frac{3}{2}l'$ for $\frac{3}{2}L$ we get the working value of θ'

$$\theta'=(100L-\tfrac{3}{2})l-\tfrac{100}{3}l'^2,$$
$$=(S-\tfrac{3}{2})l'-\tfrac{100}{3}l'^2.$$

NUMBER, DISCRETE AND CONTINUOUS.

By Dr. George Bruce Halsted, University of Texas.

PREFACE.

The modern era of the world, the scientific, dates from 1637 when Descartes published his system of conditions which we now interpret as giving to every point in a plane a distinct name consisting of two numbers, and to every pair of numbers a point. His conventions, though for his use explicable, and by him explained, as a geometric algebra operating with sects, yet get their dual power only when seen as setting up a unique one-to-one connection between number-pairs and points, so making algebra talk geometry, and inversely, geometry talk algebra. For example, the equation $Ax+By+C=0$, representing each pair of numbers which jointly satisfy the equation, pictures now an aggregate of points, which are *on* a straight line while number is discrete, but which *are* a straight line when number is continuous. Descartes perhaps never passed beyond Euclid's representation of the ratio of two magnitudes by two other magnitudes, never reached the conception of the systematic representation of the ratio of two magnitudes by one magnitude, the primitive form of continuous number. Newton makes this great step explicitly and consciously. At the beginning of his Arithmetica universalis he says:

"Per Numerum non tam multitudinem unitatum quam abstractam quantitatis cujusvis ad aliam ejusdem generis quantitatem quae pro unitate habetur rationem intelligimus. Estque triplex; integer, fractus, et surdus: *Integer* quem unitas metitur, *Fractus* quem unitatis pars submultiplex metitur, et *Surdus* cui unitas est incommensurabilis.

* * * * * . . . . . .

Quantitates vel Affirmativae *sunt seu majores nihilo, vel* Negativae *seu nihilo minores.*"

Here we have at once the whole continuous system of real number, containing not only the absolute negative, but the general irrational, for notice that here a "surd" is not a 'root', but the abstract ratio of any possible sect incommensurable with the unit sect. We may readily prove rigidly that ratios combine according to the same laws as natural numbers.

Following Euclid, we know that any ratio may be changed into one with a given second term. If then x equals the ratio of the sect A to the sect D, and $y = B:D$, then the ratio $(A+B):D$ equals $x+y$, the *sum* of the ratio x and y, a magnitude independent of D. This addition obeys the same laws as that for natural numbers, and the inverse $x-y$ is always possible and determinate, if $x>y$.

That Euclid's well-known composition of ratios obeys the same laws as ordinary multiplication of natural numbers and fractions, I have shown on page 205 of my Elements of Geometry, and that the same holds for division follows from the problem on page 203, 'To alter a given sect in a given ratio', which is nothing but dividing the sect by the ratio.

For any one who is willing to base the continuity of the real number-system on the continuity of space, for any one who is satisfied to say of the entire system of real numbers, that, inasmuch as it contains an individual number to correspond to every individuhl point in the continuous series of points forming a right line, it is continuous, this ratio method would seem to be the only logical one. The defining of numbers by regular sequences, the use of series, the theory of limits, and various new mathematical conceptions have been employed by Weierstrass, G. Cantor, and Dedekind in establishing three independent and equally cogent theories which should prove the continuity of number without borrowing it from space. I do not know of the existence of either of these demonstrations in English. Fine's Number-System starts G. Cantor's theory, but does not get as far as either of Cantor's fundamental concepts' "zusammenhængend und perfect", but instead is content to get continuity from the line. Upon this procedure Dedekind is particularly severe. He keeps his theory wholly pure from any admixture of measureable magnitudes, and maintains that for a great part of the science of space the continuity of its forms is not a necessary presupposition, and gives the following example: If we take any three non-collinear points with only the specification, that the ratios of the sects AB, AC, BC are algebraic numbers, and consider as present in space only those points M, for which the ratios of AM, BM, CM to AB are likewise algebraic numbers then the space consisting of these points is throughout discontinuous; [it lacks all points D for which a ratio, as AD, to AB in a transcendent number such as π or e]; yet despite the discontinuity, the perforation, of this space, all constructions occurring in Euclid are in it just as achievable as in perfectly continuous space. The discontinuity of this space would therefore never be noticed, never be discovered, in Euclid's science.

"Um so schöner erscheint es mir, dass der Mensch ohne jede Vorstellung von messbaren Grössen, und zwar durch ein endliches System einfacher Denkschritte sich zur Schöpfung des reinen, stetigen Zahlenreiches aufschwingen kann; und erst mit diesem Hülfsmittel wird es ihm nach meiner Ansicht möglich, die Vorstellung vom stetigen Raume zu einer deutlichen auszubilden".

CHAPTER I.

COUNTING AND NATURAL NUMBERS.

I. NUMERALS.

1. An Algebra is an artificial language composed of symbols with their laws of combination, and possessed of peculiar advantages in giving of actual relations representations which can be manipulated according to rules of operation and procedure, experimented upon to give new knowledge, according to organized processes. The first algebra was slowly formed throughout centuries, to investigate the properties of numbers.

2. In nature, each distinct thing is perceived as an individual. Each distinct thing is a whole by itself, a unit. The individual thing is the only whole, or distinct object which exists in nature. But the human mind takes like individuals together and makes of them a single whole, and names it. Thus we have made the concept a flock, a herd, a bevy, a covey, a family. These are artificial units, discrete magnitudes; the unity is wholly in the concept, not in nature; it is of human make. From the contemplation of the natural individual in re-

lation to the artificial individual spring the related ideas 'one' and 'many'. A unit thought of in contrast to 'many' as not-many, gives us the idea *one*. A 'many' composed of 'one' and another 'one' is characterized as two. A many composed of 'one' and the special many 'two' is characterized as 'three'.

Numerals applied thus each to a special kind of discrete magnitude are called *cardinal numbers*.

But if for use in picturing all special artificial units or discrete magnitudes, we make an abstract system of elements where no characteristic of any element is retained beyond its simple distinctness from all others, and give each element successively a name, 'first', 'second', 'third', etc., these would be *ordinal numbers*.

Ordinal numbers will picture a group by successively picturing separately each element in the group. A cardinal number gives a single special picture for a special group.

Each number-picture of a group is wholly abstract, in that it represents the individual existence of the elements of the group, and nothing more.

Number is a creation of the human mind, and only applies primarily to the artificial wholes created by the human mind, discrete aggregates.

3. For the transmission of these abstract conceptions the fingers formed the original apparatus, and the name of a number denoted when referring to an artificial unit, as of sheep, that a certain group of fingers would touch successively the natural units in the discrete magnitude indicated, or a certain finger stand as a symbol for the numerical characteristic of that group of natural units.

Our word *five* is cognate with the Latin quinque, Greek πὲντε, Sanskrit pankàn, Persian pendji; now in Persian penjeh or pentcha means an outspread hand. In Eskimo 'hand me' is tam ut'che, 'shake hands' is tal la'lue, 'bracelet' is tale gow'ruk, 'five' is talema.

"In Greenland, on the Orinoco, and in Australia alike, 'six' is 'one on the other hand' ".

II. COUNTING.

4. The operation of counting consists in establishing such a correspondence between two groups that to every thing or element of the one group is assigned one particular thing or element of the other.

It establishes a one-to-one correspondence of two aggregates, one of which is carried about as a standard; and if a group of things can have this correspondence with the standard group then those properties of the standard group which are carried over by the correspondence will belong to the new group.

5. The Chinese even at the present day extend the primary standard group, the fingers, by substituting for it a group of ivory balls movably strung on rods fixed in an oblong frame. With this abacus they count and perform their arithmetical calculations.

6. In many languages there are not even words for the first ten groups, so that the actual fingers are used; higher races have not only named these groups, but have extended indefinitely this system of names, and no longer count directly with their fingers, but use the names, so that the operation of counting a certain assemblage of things consists in assigning to each of them one of these numeral words, primarily an ordinal, since the cardinal word now used represents not the individual with which it is associated, but the entire group of which this individual is the last.

III. RECORDED SYMBOLS.

7. But for purposes of counting, a group of objects can be represented by a graphic picture so simple that it can be

produced whenever wanted by just making a mark for each distinct object.

Thus the marks I, II, III, IIII, picture first the groups with a permanence beyond gesture or word, and for many important purposes, one of these diagrams, though composed of individuals all alike, is an absolutely perfect picture, as accurate as the latest photograph, of any group of real things no matter how unlike.

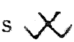
8. Such a record would not only help in getting an idea of an actual group, as a flock of sheep; but after a lapse of time, would help in recalling and accurately reproducing that idea.

Thus the shepherd who before sleeping makes such a picture of his flock, may, upon waking, use that picture to compare his flock of yesterday with his flock of to-day. The scout who makes such a picture of a band of enemies, may use it to rouse in the minds of his companions an accurate idea of what he has seen.

IV. GRAPHIC NUMERALS.

9. Each stroke of such a group may be called a unit. Each group of such units will correspond always to the same group of fingers, to the same numeral word.

10. To this primitive graphic system of numeration there is no limit, and when it becomes cumbrous the hands again suggest natural abbreviations.

The Etruscan and Roman numeral V comes probably from a picture of an open hand, and X from two V's joined thus .

11. In the Roman notation as still in use we see another and more conventional element in distinguishing IV from VI and IX from XI. This is the significant use of relative position.

12. The systematic decimal system in accordance with which, even in the times of our pre-historic ancestors, a few number names were used to build all numeral words, is sug-

gested by the procedure even at the present day of those Africans who in counting use a row of men as follows: The first begins with the little finger of the left hand and indicates, by raising it and pointing or touching, the assignment of this finger as representative of a certain individual from the group to be counted; his next finger he assigns to another individual; and so on until all his fingers are raised. And now the second man raises the little finger of his left hand as representative of this whole ten, and the first man, thus relieved, closes his fingers and begins over again. When this has been repeated ten times, the second man has all his fingers up, and is then relieved by one finger of the third man, which finger therefore represents a hundred, and so on to a finger of the fourth man, which represents a thousand, and to a finger of the fifth man, which represents a myriad.

V. THE ABACUS.

13. An advance on this actual use of fingers with a positional value depending on the man's place in the row, is seen in the almost universally occurring abacus, a rough case of which is just a row of grooves in which pebbles can slide. With most races, as with the Egyptians and Greeks, the grooves and columns are vertical like a row of men.

14. As in the written additively combined numbers of all races the greater precedes the less, so here, for races reading from left to right, the pebbles in the right-most column correspond to the fingers of the man who actually touches or checks off the individuals counted; it is the units column.

15. But in the abacus a simplification occurs. One finger of the second man is raised to picture the whole ten fingers of the first man, so that he may lower them and begin again to use them in representing individuals. Thus there are two designations for ten, either all the fingers of the first man or one finger of the second man.

The abacus omits the first of these equivalents, and so each column contains only nine pebbles.

16. And just so to-day we use nine digits and have no digit corresponding to the Roman X, for X is all the fingers of the first man, while we, like the abacus, use 10, which is one finger of the second man.

17. The use of the digits (Latin, digitus, a finger), the substitution of a single symbol for each of the first nine picture-groups, and the splendid invention, by the Hindoos, of the zero, 0, nought, cypher, made possible our present perfect positional notation for number, which the decimal point (say rather, *digital* point) empowers to run down below the units.

18. Cyphering, which thus attains an ease and facility that would have astonished a Greek or Roman, consists in combining given numbers according to fixed laws to find certain resulting numbers.

19. That the number of any finite group of distinct things is independent of the order in which they are taken, that beginning with the little finger of the left hand and going from left to right, a group of distinct things comes ultimately to the same finger in whatever order they are counted, follows simply from the hypothesis that they are distinct things. If a group of distinct things comes to say five when counted in a certain order, it will come to five when counted in any other order.

20. For a general proof of this take as objects the letters in the word *triangle* and assign to each a finger, beginning with the little finger of the left hand and ending with the middle finger of the right hand.

Each of these fingers has then its own letter, and the group of fingers thus exactly adequate is always necessary and sufficient for counting this group of letters in this order.

That the same fingers are exactly adequate to touch this same group of letters in any other order, say the alphabetical, follows because, being distinct, any pair attached to two of my fingers in a certain order can also be attached to the same two fingers in the other order.

In the new order I want *a* to be first. Now the letters *t* and *a* are by hypothesis distinct. I can therefore interchange the fingers to which they were assigned, so that each finger goes to the object previously touched by the other, without using any new fingers or setting free any already employed. The same is true of *r* and *e*, of *i* and *g*, etc. As I go to each one I can substitute by this process the new one which is wanted in its stead in such a way that the required new order shall hold good behind me, and since the group is finite, I can go on in this way until I come to the end without changing the group of fingers used in counting, that is, without altering the number, in this case eight.

21. The group of fingers exactly adequate to touch a group of objects in any one definite order is thus exactly adequate for every order. But when touching in one definite order each finger has its own particular object and each object its own particular finger, so that the group of fingers exactly adequate for one peculiar order is always necessary and sufficient for that one order. But we have shown it then exactly adequate for every order, therefore it is exactly necessary and sufficient for every order.

CHAPTER II.

THE BEGINNING OF ALGEBRA.

VI. THE SYMBOLS + AND =.

22. The natural numbers, for example, the primitive pictures I, II, III, IIII, begin with a single unit, and are changed each to the next always by taking another single unit.

The operation of incorporating this new unit into the preceding diagram may be indicated by a symbol first used in the 15th century, a little Maltese cross (+) which is read by the Latin word *plus*, and called the plus sign.

23. A number is said to be *equal* to, or the same as, a number otherwise expressed, when their units being counted come to the same finger, the same numeral word. The symbol =, read *equals*, is called the sign of equality, and takes the part of verb in this symbolic language. It was invented by an Englishman, Robert Recorde, who published it in 1557, some say 1540. Equality is a mutual relation always invertable. An algebraic sentence using this verb is called an equation.

Thus we may write

$$\mathrm{I}=1.$$
$$\mathrm{II}=1+1=2.$$
$$\mathrm{III}=1+1+1=2+1=3.$$
$$\mathrm{IIII}=1+1+1+1=2+1+1=3+1=4.$$

VII. INEQUALITY.

24. When the process of counting the units of one number simultaneously one-to-one with the units of a second number ends because no unit of the second number remains uncounted, but the units of the first number are not all counted, then the first number is said to contain more units than the second number, and the second number is said to contain less units than the first.

If a number contains more units than a second, it is called *greater* than this second, which is called the *lesser*.

By adding units to the lesser of two natural numbers we can make the greater.

25. Thomas Harriot, (1560–1621), devised the symbol >, published 1631, read 'is greater than', and called the sign of inequality.

Since the result of counting is independent of the order of the individuals counted, therefore of two natural numbers the one is always greater than, equal to, or less than the other.

Without knowing the number n, we can write either $n>5$ or $n=5$ or $5>n$.

VIII. PARENTHESES.

26. When we can get a third number from two given numbers by a definite operation, the two given numbers joined by the sign for the operation and enclosed in parentheses may be taken to mean the result of that combination.

The result can now be again combined with another given number, and so we may get combinations of several numbers, though each operation is performed only with two. Thus $(1+1)+1=3$.

Parentheses indicate that neither of the two numbers enclosed, but only the number produced by their combination, is related to anything outside the parentheses.

With the understanding that the primary view of any chain of operations is that the operations are to be carried out successively from left to right, parentheses (first used by Albert Girard, 1629) may often be omitted without ambiguity.

27. The representation of one number by others with symbols of combination and operation is called an expression.

By enclosing it in parentheses, any algebraic expression however complex, in any way representing a number, may be operated upon as if it were a single symbol of that number.

If an expression already involving parentheses is enclosed in parentheses, each pair, to distinguish it, can be made different in size or shape.

The three most usual forms are the parentheses (, the bracket [, and the brace $\{$.

In translating from Algebra into English, (should be called *first parenthesis*, and) second parenthesis; [first bracket,] second bracket; { first brace, } second brace.

IX. SUBSTITUTION.

28. No change of resulting value is made in any expression by substituting for any number its equal however expressed. From this it follows that two numbers each equal to a third are equal to one another. This process, putting one expression for another, *substitution*, is the most primitive yet the most important proceeding of algebra. A single symbol may be substituted for any algebraic expression whatever.

29. Permutation consists in a simultaneous carrying out of mutual substitution, interchange.

Thus a and b in an expression, as abc, are permuted when they are interchanged, giving bac.

More than two symbols are permuted when each is replaced by one of the others, as in abc giving bca or cab.

CHAPTER III.

THE TWO DIRECT OPERATIONS.

X. ADDITION.

30. Suppose we have two natural numbers in their primitive form, as III and IIII; if we write all these units in one row we get another natural number; and this process of putting the two groups together to make a single group, of increasing the one

group by the other, is called *addition.* Addition is such a taking together of two numbers that the units preserve their respective independence, just as objects in the taking together involved in counting them.

31. The result of addition is called a *sum*, and is attained by a repetition of the operation of forming a new group from an old by taking with it one more unit.

Thus the sum of three and two is $[(3+1)+1]$, and this is what is meant by $3+2$, so that $3+2=[(3+1)+1]$.

32. If given numbers are written as sums of units, e. g. (exempli gratia), $2=1+1$, $3=1+1+1$, the result of adding then is obtained by writing together, joined by the plus sign, these rows of units. Here it is $1+1+1+1+1=5$.

To express the addition of two and three we connect by + the parts set down in order each expressed as a whole; thus $(1+1)+(1+1+1)$, and the explanation of this expression, or the definition of the sum is given by the equation

$$(1+1)+(1+1+1)=1+1+1+1+1.$$

Since number is independent of the order of counting, therefore in any natural number expressed in its primitive form, as IIII, the permutation of any pair of units produces neither visible nor real change. The units of numeration are completely interchangable. Therefore we may say, adding numbers is finding one number which contains in itself as many units as the given numbers taken together.

33. In defining addition, we need make no mention of the order or the groups in which the given numbers are taken together to make the sum.

A sum is independent of the order of adding. $2+3=3+2$. A sum is independent of the grouping of its parts. $(4+2)+3=4+(2+3)$. For a change in the order or the grouping of the parts added is only a change in the order or the grouping of the units, which change is without influence when all are counted together.

34. To write wholly in algebra that addition is an operation unaffected by permutation or grouping of the parts added, though applied to any numbers whatsoever, we cannot use numerals, since numerals are always absolutely definite.

But if, following Vieta, 1579, we use letters as general symbols to denote numbers left otherwise indefinite, we may write a to represent the first number not only in the sum $2+3$, but in the sum $4+2$, and in the sum of any two numbers. Taking b for a second number, the algebraic sentence $a+b=b+a$ is a statement about all numbers whatsoever. It says, addition is a *commutative* operation.

35. In a sum of units, brackets inserted anywhere produce no change. The general statement $(a+b)+c=a+(b+c)$ says, addition is an *associative* operation.

XI. FORMULAS.

36. For a sum of three numbers the associative and commutative laws of liberty give the following six equivalent expressions,

$$a+b+c=b+c+a=c+a+b$$
$$=a+c+b=b+a+c=c+b+a.$$

37. Equalities like the preceding have to do only with the very nature of the operations involved, and not at all with the particular numbers operated with.

Such an equation is called a *formula*.

38. A formula is characterized by the fact that for any letter in it any number whatsoever may be substituted without destroying the equality or restricting the values of any other letter. In a formula a letter as symbol for any number may be replaced not only by any digital number, but also by any other symbol for a number whether simple or compound, in the last case bracketed. Since $a+b=b+a$, therefore $(a+c)+b=b+(a+c)=a+b+c$.

Thus from a formula we can get an indefinite number of formulas and special numerical equations.

39. Each side or member of a formula expresses a method of reckoning a number, and the formula says that both reckonings produce the same result.

40. A formula translated from symbols into words gives a rule.

As equality is a mutual relation always invertable, a formula will usually give two rules, since its second number may be read first.

41. Two or more formulas sometimes combine to give a single rule, thus

To sum any set of numbers it is indifferent in what order the given numbers are added together.

42. By definition, from the inequality

$$a > b$$

we know that a could be obtained by adding units to b. Calling this unknown group of units u, we have

$$a = b + u.$$

Inversely, if $a = b + u$

then $a > b$: that is,

a sum of natural numbers is always greater than one of its parts.

But we have proved

$$a + (b + u) = (a + b) + u,$$

and

$$(a + u) + b = (a + b) + u,$$

therefore

$$a + (b + u) > a + b,$$

$$(a + u) + b > a + b\text{: that is,}$$

a sum changes if either of its parts changes.

A sum increases if either of its parts increases.

XII. MULTIPLICATION.

43. Sums in which all the parts are equal frequently occur. Such additions are laborious and liable to error.

But such a sum is determined if we know one of the equal parts and the number of parts. The operation of combining these two numbers to get the result is called *multiplication*; the result is then called the product.

The part repeated is called the multiplicand, and the number which indicates how oft it occurs is called the multiplier.

44. In forming a product, the multiplicand is taken once for each unit in the multiplier. To multiply consists in doing with the multiplicand what is done with the unit to form the multiplier.

Following Wm. Oughtred (1631), we use the sign $\times$ to denote multiplication, writing it before the multiplier but after the multiplicand.

Thus 1×10, read one multiplied by ten, or simply one by ten, stands for the product of the multiplication of 1 by 10, which by definition equals ten. The multiplication sign may be left out when the product cannot reasonably be confounded with anything else, thus $1a$ means $1 \times a$, read one by a, which by definition equals a. From our definition also $a \times 1$, that is a multiplied by 1, must equal a.

46. Multiplication of a number by a number is commutative.

Multiplier and multiplicand may be interchanged without altering the product.

1 1 1 1 1 For if we have a rectangular array of a rows
1 1 1 1 1 each containing b units, it is also b columns each
1 1 1 1 1 containing a units. Therefore $b \times a = a \times b$.

47. Taking apposition to mean successive multiplication, for example,

$$abcde = \left\{ [(ab)c]d \right\} e,$$

calling the numbers involved *factors*, and the result their product, we may prove that commutative freedom extends to any or all factors in any product.

For changing the order of a pair of factors which are next one another does not alter the product.

$$abcd = acbd.$$

$a\ a\ a\ a\ a$ For c rows of a's, each row containing b of them,
$a\ a\ a\ a\ a$ is b columns of a's, each column containing c of
$a\ a\ a\ a\ a$ them.

So c groups of ab units come to the same number as b groups of ac units.

Consequently d groups of abc units are the same as d groups of acb units.

This reasoning holds, no matter how many factors come before or after the interchanged pair. For example,

$$abcdefg = abc\ ed\ fg,$$

since in this case the product abc simply takes the place which the number b had before. And e rows with d times abc in each row come to the same number as d colums with e times abc in column.

It remains only to multiply this number successively by whatever factors stand to the right of the interchanged pair.

It follows therefore that no matter how many numbers are multiplied together, we may interchange the places of any two of them which are adjacent without altering the product.

But by repeated interchanges of adjacent pairs we may produce any alteration we choose in the order of the factors.

This extends the commutative law of freedom to all the factors in any product.

48. To show with equal generality that multiplication is associative, we have only to prove that in any product any group of the successive factors may be replaced by their product.

$$abcdefgh = abc(def)gh.$$

By the commutative law we may arrange the factors so that this group comes first.

Thus $abcdefgh = def\,abcgh$.

But now the product of this group is made in carrying out the multiplication according to definition.

Therefore $abcdefgh = defabcgh = (def)abcgh$.

Considering this bracketed product now as a single factor of the whole product, it can, by the commutative law, be brought into any position among the other factors, for example, back into the old place; so

$$abcdefgh = defabcgh = (def)abcgh = abc(def)gh.$$

XIII. THE DISTRIBUTIVE LAW.

49. Multiplication combines with addition according to what is called the distributive law. Instead of multiplying a sum and a number we may multiply each part of the sum with the number and add these products.

$$a(b+c) = (b+c)a = ab+ac.$$

$$4 \times 5 = 4(2+3) = (2+3)4 = 2 \times 4 + 3 \times 4 = 5 \times 4.$$

Four by five equals five by 4, and four rows of $(2+3)$ units may be counted as four rows of two units together with 4 rows of 3 units.

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

As the sum of two numbers is a number, we may substitute $(a+b)$ for b in the formula

$(b+c)d = bd+cd$, which thus gives

$$[(a+b)+c]d = (a+b)d+cd = ad+bd+cd.$$

So the distributive law extends to the sum of however many numbers.

The terms 'distribute' and 'commutative' were introduced by Servois in 1813.

Rowan Hamilton in 1844 first explicitly stated and named the 'associative' law.

51. Since

$$a(b+c) > ab \quad \text{and}$$
$$(a+b)b > ab, \quad \text{therefore}$$

a product changes if either of its factors changes.

A product increases if either of its factors increases.

CHAPTER IV.

THE TWO INVERSE OPERATIONS.

XIV. INVERSION.

52. In the preceding direct operations, in addition and multiplication, the simplest problem is, from two given numbers to make a third.

If a and b are the given numbers, and x the unknown number resulting,

$$x = a + b$$
$$x = a \times b, \text{ according to the operation.}$$

An *Inverse* of such a problem is one where the previously sought number is given, and also one of the others, to find the third. The operation by which such a problem is solved is called an inverse operation.

To invert in algebra is like inverting in geometry where there are two parts in the hypothesis and one in the conclusion. The conclusion taken with either part of the original hypothesis gives the hypothesis of an inverse.

53. Since the two parts of a sum, as also the two factors of a product, in accordance with the commutative law, can be interchanged, so the inverse problem is the same whichever of the two numbers is sought, since we may make the first number the second without changing the result of the direct operation.

XV. SUBTRACTION.

54. Suppose we are given a sum which we designate by a, and one of its parts, say b, to find the other part, which, yet unknown, we represent by x.

Since the sum of the numbers b and x can also be expressed as $b+x$, we have the equation $x+b=a$.

But this equation differs in kind from the literal equations heretofore used.

It is not a formula, for any digital number substituted for one of these letters restricts the value permissible for the others.

Such an equation is called a *synthetic* equation.

55. The inverse problem for addition now consists just in this,—to *solve* the synthetic equation $b+x=a$, when a and b are given; in other words, to find a definite number which placed as value for x will satisfy the equation, that is which added to b will give a.

56. If the operation by which from a given sum a and a given part b we find a value for x is called *from* a *subtracting* b, then, using the *minus* sign ($-$) to denote subtraction, we may write the result $a-b$, read a minus b.

57. We may get this result, remembering that a number is a sum of units, by pairing off every unit in b with a unit in a, and then counting the unpaired units. This gives a number which added to b makes a.

The expression $a-b$ is called a remainder.

The term preceded by the minus sign is called the subtrahend.

By definition, $a-b+b=a,$

and also $b+(a-b)=a,$

or $b+a-b=a.$

(*To be continued.*)

GEOMETRIC INVERSION.

By Annie L. MacKinnon, Lawrence, Kansas.

1. Geometric inversion is a method of transformation by means of reciprocal radius vectors. Let O be a fixed point and P and P' two points on a line through O; P and P' are said to be inverse when $OP.OP'=M$, M being any constant. It is convenient for algebraic and geometric purposes to take $OP.OP'=1$.

2. Geometrically the operation of inversion may be thus represented. Draw a unit circle with center O. Let P be any point without the circle; join OP; from the point P draw tangents to the unit circle. M and N being the points of tangency; draw MN. The point P' at which MN intersects OP is the inverse of the point P, since $OP.OP'=1$ or $OP'=\frac{1}{OP}$ or $r'=\frac{1}{r}$.

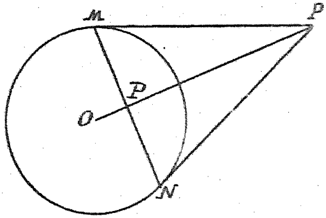


Fig. 1.

And conversely, the point P is the inverse of the point P'. It may be noticed that every point without the circle inverts into a point within the circle and vice versa.

3. In order to determine the transformation of the various figures or systems of points, it will be convenient to have expressions for the relations between the rectangular co-ordinates of the two points P and P′. From the relations of rectangular and polar co ordinates, we have the equation

(1) $$x' + y' = r' (\cos \theta + \sin \theta).$$

and since $r' = \frac{1}{r}$

(2) $$x' + y' = \frac{1}{r} (\cos \theta + \sin \theta)$$

or (3) $$x' + y' = \frac{r}{r^2} (\cos \theta + \sin \theta)$$

substituting x and y for their polar equivalents $r \cos \theta$ and $r \sin \theta$, we obtain

(4) $$x' + y' = \frac{x + y}{r^2}$$

and since $r^2 = x^2 + y^2$

(5) $$x' + y' = \frac{x + y}{x^2 + y^2}$$

$$\therefore \quad x' = \frac{x}{x^2 + y^2} \text{ and } y' = \frac{y}{x^2 + y^2} \qquad (6)$$

and conversely

(7) $$x = \frac{x'}{x'^2 + y'^2} \quad y = \frac{y'}{x'^2 + y'^2}.$$

THE INVERSION OF STRAIGHT LINES.

4. Let us examine the general equation of the straight line.

(1) $$Ax + By + C = 0$$

substituting formulae (7) of the last paragraph

$$(2) \qquad \frac{Ax'}{x'^2+y'^2}+\frac{By'}{x'^2+y'^2}+C=0$$

or

$$(3) \qquad Ax'+By'+C(x'^2+y'^2)=0$$

which is the equation of a circle through the origin. Let the circle (Fig. 2) with center O be the unit circle and AB any straight line. The circle with center C is the inverse of the line AB. Since the circle will be symmetrical with respect to OX, which is the perpendicular to AB, the diameter of the circle may be measured on OX and is the reciprocal of OP. If the line AB be tangent to the unit circle, the reciprocal of OP=1 and the diameter of the inverse circle is unity. If the line AB intersect the unit circle, the reciprocal of OP is >1 and the diameter of the inverse circle is >1. If the line AB lie without the unit circle, the reciprocal of OP is <1 and the diameter of the inverse circle is < 1. The greater the distance OP, the smaller the diameter of the inverse circle; if OP=∞, the diameter of the inverse circle $=\frac{1}{\infty}=0$, and the inverse circle is a point; and if OP=0, the diameter of the inverse circle $=\frac{1}{0}=\infty$, and the inverse circle is a straight line. We may conclude that every straight line inverts into a circle passing through the origin; unless the line itself passes through the origin, in which case the line remains unchanged, although the order of its points is changed. From the above it may be seen that a system of parallel lines inverts into a system of tangent circles passing through the origin whose centers are on

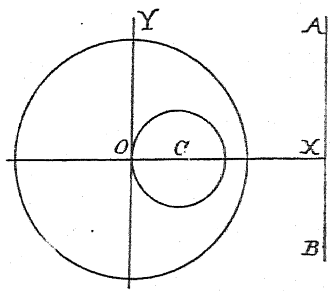


FIG. 2.

a line perpendicular to the parallel lines; and a pencil of lines inverts into a system of intersecting circles passing through the origin.

THE INVERSION OF CIRCLES.

5. Substituting formulae 3 . . . (7) in the general equation of a circle

(1) $$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

we obtain

(2) $$\frac{A(x'^2+y'^2)}{(x'^2+y'^2)^2} + \frac{2Gx' + 2Fy'}{x'^2+y'^2} + C = 0$$

or (3) $$A + 2Gx' + 2Fy' + C(x'^2+y'^2) = 0$$

which is still the equation of a circle. If the circle pass through the origin, C=0 and F=0 and the equation of the circle is

(4) $$x^2 + y^2 + 2Gx = 0$$

which becomes upon inversion

(5) $$\frac{x'^2}{(x'^2+y'^2)^2} + \frac{y'^2}{(x'^2+y'^2)^2} + \frac{2Gx'}{(x'^2+y'^2)} = 0$$

or

(6) $$1 + 2Gx' = 0$$

which is the equation of a straight line, (converse of the inversion of a straight line).

The equation of a circle with the origin as center

(7) $$x^2 + y^2 = r^2$$

inverts into

(8) $$\frac{x'^2}{(x'^2+y'^2)^2} + \frac{y'^2}{(x'^2+y'^2)^2} = r^2$$

(9) $$x'^2 + y'^2 = r^2(x'^2+y'^2)^2$$

(10) $$x^2 + y^2 = \frac{1}{r^2}$$

the equation of a circle concentric with the given circle and whose radius is $\frac{1}{r}$. It is evident that if a circle cut the unit circle in two points, the inverse circle will pass through the same two points.

THE ANGLE OF THE INVERSE CURVE.

6. Let A and B be two points (Fig. 3) on any curve, and A′ and B′ corresponding points on the inverse curve.

$$OA.OA'=1$$
$$OB.OB'=1$$
$$\therefore\ OA.OA'=OB.OB',$$
or
$$\frac{OA}{OB'}=\frac{OB}{OA'}$$

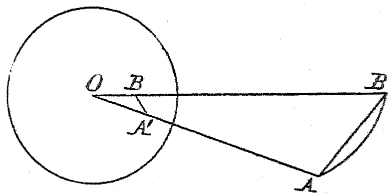


FIG. 3.

and $\angle O$ is common to the $\triangle$'s OAB and OA′B′; therefore the $\triangle$'s are similar and $\angle OB'A'=\angle OAB$,

$$\angle OBA=\angle OA'B'$$

and
$$\angle AA'B'=\angle A'B'O+\angle O.$$

If the point B approach A the cord AB will have for its limit the tangent to the curve at A, and the chord A′B′ will be approaching at its limit the tangent at the point A′; and $\angle O$ will become zero and $\angle AA'B'=\angle A'AB$. We may conclude that a radius vector cutting two inverse curves, makes angles with the tangents at the points of intersection which are equal but measured in opposite directions; and also that if two curves meet, their inverse curves meet at an angle equal to the aagle of the first curves and measured in an opposite direction.

7. It is now evident that if a given circle cut the unit circle orthogonally at the points A and B, the inverse circle will cut the unit circle orthogonally at A and B; and that in order to satisfy this condition the inverse circle must coincide with the given circle. Conversely, every circle unchanged by inversion cuts the unit circle orthogonally, unless it be the unit circle itself.

8. If a circle pass through a given point and the inverse of that point, the circle and its inverse circle will coincide; since the circles will have four points in common, the given point and its inverse and the two points of intersection on the unit circle. Any system of circles through two points and not passing through the origin inverts into another system of circles through two points. Any system of circles tangent to the unit circle inverts into a system of circles having internal contact with the unit circle at the point of tangency of the given system.

INVERSION OF CONIC SECTIONS.

9. Substituting formulae 3 . . (7) in the general equation of the second degree,

(1) $$ax^2+2bxy+by^2+2gx+2fy+c=0$$

we obtain

(2) $$ax^2+2bxy+by^2+(2gx+2fy)(x^2+y^2)+c(x^2+y^2)=0$$

the equation of a bicircular, nodal quartic.

Transferring the origin to a point on the curve, the equation becomes,

(3) $$ax^2+2bxy+py^2+2gx+2fy=0$$

and substituting formulae 3 . . . (7)

(4) $$ax^2+2bxy+py^2+(2gx+2fy)(x^2+y^2)=0,$$

an equation of the third degree, the equation of a circular, nodal cubic.

Since in the equation of the inverse curve the absolute term and the terms of the first degree vanish, the inverse curve has a double point. This double point is an acnode, crunode or cusp, according as the original curve is an ellipse, hyperbola or parabola.

10. THE ELLIPSE.—The equation of the ellipse referred to the axes

(1) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

becomes upon inversion

(2) $$b^2x^2+a^2y^2=a^2b^2(x^4+2x^2y^2+y^4),$$

one of the foci being taken as origin, the equation of the ellipse becomes

(3) $$b^2(x-c)^2+a^2y^2=a^2b^2$$

inverting

(4) $$b^2x^2-2b^2cx(x^2+y^2)+a^2y^2=(a^2b^2-b^2c^2)(x^2+y^2)$$

the equation of a *limaçon.*

11. THE HYPERBOLA.—The equation of the hyperbola

(1) $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

being similar to that of the ellipse excepting in the sign of the coefficient of y will evidently invert into the same forms as the equation of the ellipse. The equilateral hyperbola whose equation is

(2) $$x^2-y^2=a^2$$

inverts into

(3) $$(x^2+y^2)^2=\frac{1}{a^2}(x^2-y^2)$$

the equation of the Lemniscate of Bernoulli.

12. THE PARABOLA.—Inverting the equation of the parabola with vertex as origin

(1) $$y^2=4px$$

we obtain the equation of the *Cissoid* of Diocles.

(2) $$y^2=4px^3+4pxy^2.$$

transfering the origin to the focus the equation of the parabola becomes

(3) $$y^2=4px+4p^2$$

inverting

(4) $$y^2=4px^3+4pxy^2+4p^2x^4+8p^2x^2y^2+4p^2y^4$$

the equation of the *cardioid*.

INVERSE OF HIGHER CURVES.

13. Considering the general equation of a cubic

(1) $$xy^2=Ax^3+3Bx^2+3Cx+D$$

and inverting, we obtain

(2) $$xy^2=Ax^3+3Bx^2(x^2+y^2)+3Cx(x^2+y^2)^2+D(x^2+y^2)^3$$

which is an equation of the sixth degree. Inverting the general equation of the cubic with the origin on the curve ($D=0$) we obtain

(3) $$y^2=Ax^2+3Bx(x^2+y^2)+3(x^2+y^2)^2$$

an equation of the fourth degree.

Several special forms of the cubic invert into curves of lower order than the sixth. The inverse curves of the conics assume upon the second inversion the original conic form or become curves of the fourth degree.

14. The transformation by inversion of a curve of the nth degree will exhibit the various forms which the *higher curves* may assume. We write the general equation in the form

(1) $$u_0+u_1+u_2+u_3+\ \ldots\ +u_n=0$$

in which u_0 denotes the absolute term and u_1, u_2, u_3, etc., denote the terms of the first, second, third, etc., degrees. The inverse of a curve of the nth degree is in general a curve of the $2n$th degree, the equation assuming the form

(2) $$u_{2n}+u_{2n-1}+u_{2n-2}+\ \ldots\ +u_n=0$$

if the center of the inversion be on the curve, $u_0=0$ and the inverse curve is of the degree $2n-1$. If the center of inversion be a double point on the given curve, $u_0=0$ and $u_1=0$ and the inverse curve is of the degree $2n-2$; if the center of inversion be a multiple point of the order k, the degree of the inverse curve is of the order $2n-k$. Since a curve of nth degree can have no multiple point of higher order than $n-1$, the degree of the inverse curve cannot be less than $2n-(n-1)$ or $(n+1)$.

If the given curve of the nth degree be circular the degree of the inverse curve in each case given above is diminished by 2. The inverse of a curve of the nth degree has a multiple point of the order k at the center of inversion. If the given curve has a multiple point of order k not at the center of inversion, the inverse curve will have a multiple point of the same order.

15. Having two inverse curves with tangent circles at inverse points, the tangent circles are inverse to each other with respect to the origin of the first curves. The radius of curvature of a curve at any point is measured on an osculating circle; the radius of curvature on the inverse curve may be measured on the inverse of the osculating circle. If the osculating circle pass through the origin, it inverts into a straight line; and the point inverse to the point of osculation is a point of inflexion, since its radius of curvature is infinite as denoted by the straight line.

APPLICATIONS.

16. It is as a method of proof that inversion shows its power. In deducing the properties of higher curves from those of lower order whose properties are well known and readily found, it may be considered a useful instrument in *modern geometry*; and is superior in this respect to *projection* and *reciprocation*, both of which prove the properties of curves from those of the same order.

Properties which concern the magnitude of lines and curves are in general not transferable by inversion; but it is evident that lines through the origin furnish exceptions to this general rule, lines through the origin being unchanged by inversion. Also a certain proportion may give a set of corresponding proportionals in the inverse figure. All properties relating to the relative position of lines and curves are readily inverted; and these properties furnish sufficient material for the further development of properties and measurements on the inverse curve.

17. In the examples given below use is made of the preceding proofs without any reference to particular propositions. We first give a full explanation of one case of the transference of the properties of a conic to a higher curve by the method of inversion.

The tangents to a parabola at the extremities of a focal chord which makes an agle θ with the axis are inclined at angles of $\frac{\theta}{2}$ and $\frac{\pi}{2}-\frac{\theta}{2}$ to the chord. The *parabola* inverted with the focus as origin becomes a *cardioid* having its cusp at the origin [12+(4)]; the focal chord becomes a cuspidal chord in the cardioid retaining the inclination θ[4:6]; the tangents to the parabola invert into circles touching the *cardioid* at the extremities of the cuspidal chord and at angles of $\frac{\theta}{2}$ and $\frac{\pi}{2}-\frac{\theta}{2}$. Since the angle at which a circle meets a line may be measured by the angle made by a tangent at the point of meeting, and since the tangent of the tangent circle will be also the tangent of the *cardioid*, it is true that the *tangents to a cardioid at the extremities of a cuspidal chord which makes an angle θ with the axis are inclined at angles of $\frac{\theta}{2}$ and $\frac{\pi}{2}-\frac{\theta}{2}$ to the cuspidal chord.*

18. Examples:

(1.) There are three points on a conic whose osculating circles pass through a given point on the curve; these three points lie on a circle passing through the given point.

Inversely.—The three real points of inflexion of a circular, nodal cubic lie on a straight line.

(2.) The eight points of contact of two conics with their four common tangents lie on another conic.

Inversely.—The eight points at which two bicircular quartics having the same double point, come in contact with their four common tangent circles passing through the origin, lie on a third bicircular quartic having the same double point.

(3.) If two vertices of a triangle move along fixed right lines while the sides pass each through a fixed point the locus of the third vertex is a conic section.

Inversely.—(*a*)—If a system of three intersecting circles moves so that two of either set of the points of intersection move along two fixed lines and the circles each pass through a fixed point, the locus of the third intersecting point of the system is a bicircular quartic.

Inversely.—(*b*)—A system of three circles each passing through a fixed point and also through the point of intersection of two fixed lines, along which two of the points of intersection of the circles move; the locus of the third point of intersection is a bicircular quartic.

(4.) If A, B, C be three conics having each double contact with S, a fourth conic, and if A and B both touch C, the line joining the points of contact will pass through an intersection of common tangents.

Inversely.—Three nodal, bicircular quartics, A, B, C having double contact with a fourth quartic S of the same kind, all having common double points, and A and B touching C; the line joining the points of contact will pass through an intersection of common tangents.

(5.) The locus of the points of contact of tangents to a series of confocal ellipses from a fixed point on the major axis is a circle.

Inversely.—Each of a series of *limaçons* having a common axis and double point is touched by one of a series of circles passing through the double point and a fixed point on the axis; the points of contact are on the circumference of a circle.

(6.) If an equilateral hyperbola circumscribe a triangle, it will also pass through the intersection of the perpendiculars.

Inversely.—Given three circles through the double point of a *lemniscate*, intersecting the latter in three fixed points; three other circles through the same double point and through the same three fixed points, and at right angles to the first set intersect in a common point on the lemniscate.

HARMONIC PROPERTIES.

19. If the origin O be considered the vertex of an harmonic pencil O—ABCD, it is evident that the points A, B, C, D on the transversal AD will have corresponding inverse points on the rays of the given pencil. The inverse of the transversal AD is the circle passing through the vertex O, the diameters and chords of which circle will be cut harmonically by the rays of the pencil O—ABCD and will serve to establish an-harmonic properties in the inverse figures. Following this line of reasoning, we find by inversion that all anharmonic points on a conic will have corresponding points on a cubic or quartic, that all harmonic points on the cubic will have corresponding points on the curves of the fourth, fifth or sixth degree, etc. All lines in conics which are cut harmonically will have corresponding circles in the inverse figures and these circles will have harmonic points corresponding to those in the lines. By inversion, certain forms of these curves with

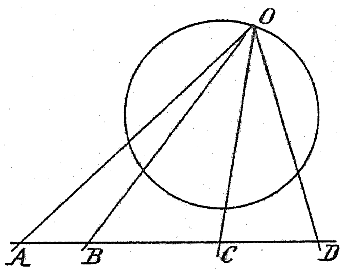


FIG. 4.

their circles and lines will become higher curves with circles showing corresponding harmonic points, etc.

But since a line in one figure will not always have a corresponding line in the inverse figures, the harmonic properties of the higher curves cannot always be obtained directly, but may be derived easily from the inverse circles corresponding to the lines in the lower curves or conics.

20. In the following examples the term "harmonic points of a circle" is used in reference to the points on a circle through which an harmonic point may be drawn. The term is similar to that of "points in involution on a circle" in the usual applications of that term.

EXAMPLES:

(1.) If three conics pass through four fixed points, the common tangent to any two is cut harmonically by the third.

Inversely.—If three nodal, circular cubics have a common double point and pass through three other fixed points, the common tangent circle through the common double point to any two of the cubics is cut harmonically by the third

(2.) A system of conics passing through four fixed points meets any transversal in a system of points in involution.

Inversely—A system of bicircular, nodal quartics having a common double point and passing through three other fixed points is cut by any transversal or circle through the double point in a system of points in involution.

(3.) Given two conics having double contact with each other, any chord of one which touches the other is cut harmonically at the points of contact and where it meets the chord of contact of the conics.

Inversely.—Two nodal, circular cubics having double contact with each other, one point of contact being a common double point, any circle through the origin touching one of them and cut by the other is cut harmonically at the points of contact and where it meets the chord of contact.

(4.) A variable chord drawn through a fixed point O to a conic subtends a pencil in involution at any point on the curve.

Inversely.—A system of circles through the double point of a nodal, circular cubic and any other fixed point is cut by the cubic in pairs of points which determine a pencil in involution.

(5.) Given four points of a conic; the anharmonic ratio of the pencil joining them to any fifth point is constant.

Inversely.—Given four circles through the double point and four fixed points of a nodal circular cubic and intersecting in any point P on the cubic, the anharmonic ratio of the pencil of tangents to the four circles at P is constant. Also any line through the double point cuts the four circles in points whose anharmonic ratio is constant.

(6.) Four fixed tangents to a conic cut any fifth in points whose anharmonic ratio is constant.

Inversely.—Four fixed circles tangent to a quartic and passing through the origin are cut by a fifth tangent circle through the origin in points whose anharmonic ratio is constant.

(7.) If A and B be two conics having each double contact with S, a third conic, the chords of contact of A and B with S, and their chords of intersection with each other meet in a point and form an harmonic pencil.

Inversely.—Given three bicircular; nodal quartics A, B and S having a common double point and A and B each having double contact with S so that the chords of contact of A and B with S pass through the common double point; then the chords of intersection of A and B also pass through the common double point and the four lines form an harmonic pencil.

21. It is now evident that the properties of conics may be extended to curves of the third and fourth degrees by the process of inversion. The inverse curves of the conics will themselves invert into the form of the original conics or into quartics. But other forms of the cubic and quartic whose properties may be obtained through a comparison with the inverse conic, will invert into still higher curves. And from the inversion of the general equation, we judge that particular forms of these curves will invert into still higher curves.

22. *The method of inversion* may be applied to *geometry* of three dimensions. All points on a given surface may be inverted and produce an inverse surface. Planes, spheres and various curved surfaces will produce upon inversion forms analogous to the inverse figures of lines, circles and conics. An interesting series of developments in *geometry* of three dimensions might be thus obtained. The transformation would be very similar to those on one plane and furnish no new elements to the theory of *geometric inversion.*

THE PNEUMATIC-HYDRAULIC SAND-LIFT.*

BY PROF. W. H. ECHOLS, UNIVERSITY OF VIRGINIA.

When sinking the caissons for the foundations of the St. Louis bridge the engineers made use of what was called the hydraulic sand-lift, or simply the sand pump, for lifting to surface the material excavated in the interior of the caisson. The construction of the machine was in principle quite simple. A pipe of certain diameter (in the present case the diameter was $3\frac{1}{2}$ inches), open at both ends, is sunk in the water until one end is at the bottom where the material is to be excavated, while the other end projects above the free level of the water surface just enough to permit of the disposal of its flow. In the lower end of the pipe or through its side near the lower end is inserted the nozzle of a smaller pipe, the direction of which is as nearly as may be in the axis of the larger. This smaller pipe is connected with a force pump at surface, which forces through the smaller pipe a flow of water under high velocity and injects it into the larger pipe at or near its lower end. The result is a flow of water out of the upper end of the larger pipe, part of which flow is the water injected by the pump and part of which is drawn into the lower end of the

*A paper read before the Philosophical Society of the University of Virginia.

flow pipe from the reservoir in which the operation takes place; and if there be sand, silt detritus of small grain, etc., in the reservoir water sucked into the lower end of the flow pipe this will be carried to surface and discharged there also.

The principle of the action of the water jet in giving motion to the column of water in the flow pipe is not so simply determined. It appears that the discharge of the jet of high velocity into the larger volume of liquid of comparatively low velocity is, by reason of the viscosity of the water, rapidly spread out laterally. This lateral spread being confined by the walls of the flow pipe serves as a sort of fluid piston through which the kinetic energy of the jet is transformed partly into a static lifting pressure on the overlying column and partly into the kinetic energy of the overflow. It is clear that the action of the pump depends upon the *kinetic energy* of the injected mass of water which is utilized through the viscosity of water. The efficiency of the machine depends upon the completeness with which the viscosity of the water permits the transforming of the energy of the jet to that of useful work in the water of the flowpipe column. This is a function of the size of the pipe and shape of the nozzle, as for instance flaring the nozzle should distribute the high velocity water of the jet more quickly and effectively over the cross section of the flow pipe.

I regret that I am without data on the application and performance of this pump, which has been frequently used in like engineering constructions since that of the St. Louis bridge. Mr. R. H. Elliott made considerable use of it in sinking the cylinders of bridge piers on the Louisville, New Orleans and Texas Ry., in Mississippi. Trautwine's Engineer's Pocket Book states, that, "With a pump pipe of 3½ inches bore, and a water jet of 150 lbs per sq. in., 20 cubic yards of sand per hour were raised 125 feet. * * * A jet of air has also been successfully used in the same way, as at the New York suspension bridge, etc."

A water jet under 150 lbs. intensity of pressure means a statical head of 345 feet and a corresponding velocity of about 150 feet per second lineal discharge of jet. The lift of 125 feet, means of course the lift of the sand through the whole length of the pipe. The lift above the free level surface is not given.

The second part of the quotation introduces the subject immediately in hand. If a jet of air be used instead of water, the discharge of sand, water and air through the flow pipe follows in a manner similar to that when water is used.

To the aparatus described below, the name Hydro-Pneumatic Lift, has been applied by Mr. Elmo G. Harris, who experimented with it while sinking the piers for the foundation of a bridge over the Arkansas River near Pine Bluff.

Mr. Harris employed an iron pipe of 3 inch bore, 20 feet in length. About 6 inches above the lower end of the pipe an inch pipe was let into the side of the 3 inch pipe and *at right angles to it.* The flow pipe was allowed to rest directly on the sand, at the bottom, with its own weight in 16 feet of water; thus the upper end of the pipe was four feet above free surface level. A flexible hose was attached to the inch pipe and an air supply driven through it by means of an ordinary force air pump which was used for supplying the air for a diver's helmet. The result was an abundant discharge of water and sand from the flow pipe, intermingled with air. As the flow pipe sunk in the sand at the bottom it was moved about from place to place over the area to be excavated.

Mr. Harris used no means of measuring the quantities of air supplied nor water and sand delivered. He merely states that he estimates the discharge to be about equal parts of sand and water.

If this be true, then as an excavator or dredge of river sift, sand, mud or any ordinary sedimentary detritus we have no superior. Its value will be found not only in the construction of bridge foundations, in dredging caissons, etc., but also in the larger operations of dredging river channels, harbor bars and the like. A valuable application in the industrial arts would be the raising of liquids through means of a pump without valves, and to which the aeration from the air used would be no injury, whereas the liquid or water injector would dilute and otherwise impair. Such might be pumps for raising, through low lifts, acids, beer, molasses, etc.

The idea of such a lift is not a new one. In Callon's Lectures on Mining delivered at the School of Mines in Paris (see English translation, Paris, 1876, pp. 307–308), in his description of Triger's method of sinking a mining shaft through very aquiferous strata at Chalonnes, by means of a pneumatic cylinder, he gives the following design for expelling water and sand from the lowest compartment (see also diagram of Triger's cylinder, Trautwine's Pocket Book, p. 648).

A pipe is run down inside the cylinder from surface to its bottom, in the lowest compartment there is a cock for the admission of air.

"* * * Besides the details given above we may mention the contrivance by means of which the pit can be kept dry, in certain cases, without requiring to increase the pressure of the air in the interior, to the whole extent due to the pressure of the water. The tube A allows the water to flow out as it accumulates and is acted on by the greater pressure of air in the shaft. The ascending column of water acts like a blowing machine, drawing in air by the cock B, which is opened to a suitable extent. The effect of this aspiration is to change the mass of the liquid into a kind of froth having a less density than water, thus allowing it to be raised to the surface where it flows out. * * Another artifice, somewhat similar to the above, consists in employing the tube not for the exit of the water which cannot escape through the surrounding ground, but for getting rid of the solid matter itself. It is possible with very running sands, to establish a current of air in the tube which carries up the sand and water together to the surface".

My attention having been called to this pump by Mr. Harris, I determined to make a series of experiments on a small scale for the sake of the theoretical interest which they might afford as well as for the practical bearing their results might have on the action of the machine on larger scale.

I propose now to present the explanation of the underlying principle of the action of the pump, the results of one series of experiments on a particular case and to exhibit by actual experiment the pump in operation.

It is evident in the beginning that the cause of the action of the pump under the air and under the water motor must be entirely different. The latter, acting through its *kinetic energy*, which it communicates to the water in the flow pipe through fluid friction, *must be injected* with high velocity; while the former acting through its *potential energy* alone (velocity of injection plays no part, at least no appreciative part, in the working of the pump in so far as its kinetic energy is concerned) need only to be delivered in sufficient quantity at a certain depth below free surface level.

Referring to Fig. 1., where are represented seven different stages of the action of the pump; consider, first, pipe I, in which we have a vertical cylindrical pipe open at both ends submerged until the lower end is d units below the surface of the water in the reservoir, while the upper end stands h units above that level.

Insert a bubble of air (whose volume is something greater than that of a sphere whose radius is the pipe-radius) in the tube as represented by 1 in pipe II. The pressure on the base of pipe II from the reservoir side is the same as that on the base of pipe I, in order therefore that equilibrium may exist in II, there must be the same weight of water in II that there is in I (less the weight of the air in the bubble), provided no flow of water takes place around the bubble.

The weight of an air bubble will be neglected in comparison with the weight of an equal volume of water, since the latter is 773 times as heavy as the air. This being the case, the free

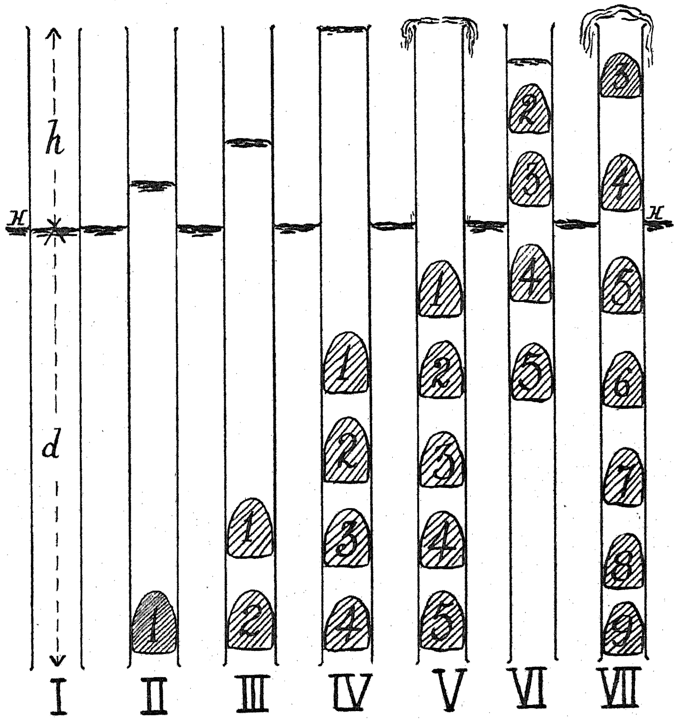


FIG. 1.

level in tube II must stand above the free level H in the reservoir by an amount such that the volume of water in II which is above H, is equal to the volume of the bubble 1.

If the bubble filled the tube completely it would remain stationary where it is, in equilibrium, transmitting the pressure unchanged between the liquid above and below it. However, the bubble never completely fills the tube, but is always bounded by a thin film or skin of liquid which lines the pipe around the bubble, so that there is always a thin liquid communication between the water above and that below the bubble. This being the case, there will be a slow transfer through this communication of the liquid above to that below the bubble, in the effort to restore the equilibrium of the water columns, which is accompanied by a corresponding slow subsidence in the free level in the tube, which in turn gives rise to an unbalanced upward pressure on the bubble equal to the weight of water transferred from above to below it. The bubble will then slowly move up the tube. The loss by leakage of the liquid around the bubble is partly restored by the expansion of the bubble as it rises, which restores the free level in the tube in some degree. This leakage is very small, under the circumstances which we are now considering, for glass tubes, as the sequel will show. It plays no part appreciably in explaining the subsequent action of the pump and is introduced now to merely account for the fact that the bubble will slowly creep up the tube.

So soon as 1 is out of the way and has reached a position such as it has in pipe III, insert another bubble 2, with the result that the volume of water which is above H in pipe III is now equal to the volumes of bubbles 1 and 2. Continue to insert bubbles 3, 4, etc., until as in pipe IV the volume of the bubbles in the pipe is equivalent to the volume of the pipe of length h, when the free level of the liquid in the pipe is at its upper end or the pipe stands full.

The insertion now of another bubble 5 causes a discharge of water from the summit of the pipe whose volume is the volume of bubble 5. The displacement of this water, leaves the contents of the pipe unbalanced and the whole pipe column is driven upward with a constant pressure equal to the weight of a volume of water equal in volume to bubble 5; so that all the water in the pipe above bubble 1 is driven out with an accelerated velocity, until finally bubble 1 itself escapes, when equilibrium will be restored by the free level in the pipe VI standing above H at such a height that the volume of water between these levels is equal to that of bubbles 2, 3, 4 and 5 together.

If now instead of ceasing with bubble 5, we continue to supply with more or less regularity bubbles 6, 7, 8, 9 and so on, until there are at all times n bubbles in the pipe, the volume of any m of which is equivalent to the capacity of a length of pipe h; then the contents of the pipe are driven upward at all times by a constant pressure which is represented by the weight of a volume of water equivalent to the volume of the remaining $n-m$ bubbles.

With the escape of each bubble from the top of the pipe there is a break in the water continuity of the flow, but if the supply of bubbles be uniform and steady and such that in the tube the distance from the bottom of one bubble to the bottom of the next is a divisor of $d+h$ the length of the pipe; then the escape of the nth bubble and the insertion of the $2n$th are simultaneous and there is no appearance of intermittency in the discharge of the water.

If the volume of the bubble should not be so large as that of a sphere whose radius is the pipe-radius, or does not fill the bore, then the leakage around the bubble takes place rapidly and the level of the liquid in the pipe is quickly restored to that of the reservoir. Indeed if the bubble is small with respect to the pipe there is no appreciable lifting of the level of the liquid in the pipe, the rise of the bubble is rapid and takes place in a manner more nearly approximating to that of a bubble in a large reservoir.

Returning now to consideration of the pump at work as in Fig. 1, pipe VII; let there be delivered at the depth d, uniformly, A cubic units of air in any definite length of time T, and a corresponding delivery of W cubic units of water at the height h in the same time. Let V be the uniform linear velocity of flow in the pipe and T the duration of flow in seconds.

The work done in delivering the A volumes of air at depth d, is restored in lifting W volumes of water through height h and giving it the velocity V, and in addition, part of the potential energy of the delivered air is dissipated through the loss of water by leakage around the bubbles, and part of it is consumed in overcoming the frictional resistances to the motion of the fluids through the pipe. If we represent the loss of potential energy through leakage by l and the work done in overcoming friction by r, we have, if w be the weight of a unit volume of water,

$$wAd = wWh + \frac{1}{2}\frac{wW}{g}V^2 + l + r.$$

Where

$$V = \frac{A + W}{\Omega T},$$

Ω being the area of cross-section of the pipe. The fact that the bubbles of air have velocity relative to the water makes this value for V a *little* too large in the ordinary cases of steady flow. The term r is so very small that it need not be considered with respect to l which is in general not small.

The efficiency of the machine as a pump is

$$E = \frac{W}{dA}(h + h_v),$$

h_v being the velocity head.

The series of experiments, to the results of which I invite your attention, consist of some fifty determinations of the quantities A, W and T for a particular pipe, the history of which is as follows.

The unit is the centimetre, linear and cubic. The time was determined with a stop watch readig to $\frac{1}{5}$th of one second.

The flow pipe was a glass tube, of the general shape of the tube marked 4 in Fig. 2, the diameter and cross-sectional area of which were determined by filling the tube with water for 97.9 c. of its length, measuring the volume of the water and computing the diameter and section. The amount of water used was $39\frac{1}{2}$c. c., giving a section of 0.4035 sq. c., a diameter of 0.7168 c. The upper end of the tube was turned to a wide flare and the lower end slightly flared to a tapering funnel to better receive the air bubbles. The length of the pipe $h+d$ was 60 c.

The most complete set of determinations were made with $h=12$ c. and $d=48$ c., or one-fifth of the pipe above surface level.

The air was delivered at the depth of 48 c. uniformly and steadily by aid of the auxiliary apparatus exhibited in Fig. 2., which consisted of a pair of graduated flasks A and B, which were scaled to read cubic centimetre contents. A was filled with water and tightly corked. Through the cork passed two tubes, 1 opening freely in the air outside and also inside at a depth c below the level of the water in A, 2 passing from the bottom of A through the cork down through the cork of B, into B, opening freely. Tube 3 opens freely in B at one end and freely in the reservoir of water R at the other end just under the mouth of the flow pipe. The axis of delivery of 3 being horizontal so as to avoid even the appearance of *jet* action being considered. The tube 2 acting as a syphon out of A has a rubber joint provided with one or more set screw clamps which may be used to regulate the aperture at will or cut it off altogether if desirable and thus giving a means of instantly starting or stopping the flow. During the measurement, the discharge from pipe 4 was caught in a graduated flask, and an auxiliary water supply kept the level of the water in the reservoir at a constant elevation. The capacity of A and that of B was something over 2000 c.c.

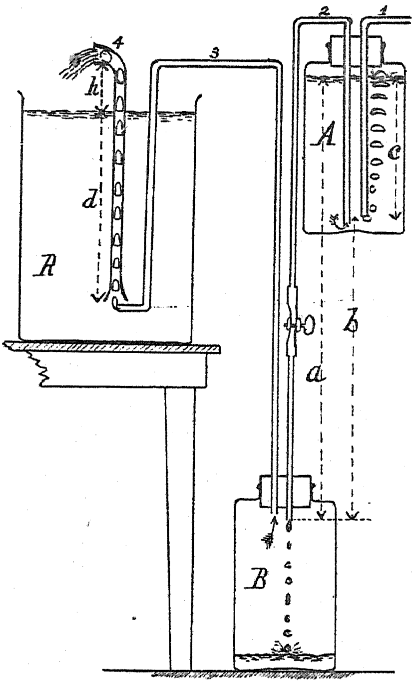


FIG. 2.

It is evident that when the syphon 2 is running and the air being delivered in R, the intensity of pressure throughout the air space in B is d hydrostatic units, so that the water flows out of A into B under a static head of pressure

$$a - c - d \text{ or } b - d,$$

which is constant, and in this particular experiment was 104 c. Therefore the number of c.c of air delivered under the flow pipe 4 is equal to the number of c.c of water passed from A to B (the change in air volume being inappreciable). The velocity of this uniform flow was regulated by the clamp on tube 2.

An experiment consisted in starting the flow until all the tubes were full, clamping the rubber connection in 2, reading A and B, springing the watch and releasing the tube at the same instant. The flow of 4 was caught. The end of the experiment consisted in merely clamping the tube and stopping the watch simultaneously, and the readings being made at leisure. These details are entered into with minuteness in order to show how nearly the accuracy of the results may be depended on.

In the following table the column A contains the number of c.c of air delivered per second; as computed from the quantities Q_a Q_w which represent the total number of c.c of air delivered and water discharged during T seconds respectively.

TABLE I.

12 cm. out and 48 cm. in the water.

No.	A	W	Q_a	Q_w	T
1	18.7	20.6	1350	1480	72
2	19.2	20.5	1500	1600	78
3	19.3	20.0	1450	1500	75
4	19.5	19.9	1600	1630	82
5	19.0	19.9	1600	1670	84
6	19.0	19.6	1500	1550	79
7	16.8	19.6	1425	1660	85
8	14.6	18.7	1500	1920	103½
9	15.2	18.6	1550	1900	102½
10	18.7	18.3	1400	1370	75
11	12.1	17.0	1600	2130	125
12	11.0	16.4	1500	2200	136
13	7.6	15.0	1350	2600	178
14	7.9	15.0	1350	2530	170
15	7.1	14.0	650	1270	91
16	6.2	13.6	550	1210	89½
17	5.9	13.4	500	1140	85½
18	5.7	13.4	500	1180	87
19	6.2	13.3	1000	2140	161
20	6.0	13.3	1000	2200	166
21	6.5	13.1	1250	2530	193
22	6.4	12.6	1250	2450	193½
23	5.5	12.1	1600	3510	291
24	5.3	11.4	575	1330	108
25	5.0	11.2	400	880	79
26	3.7	8.7	600	1400	161
27	3.1	8.2	900	2350	288
28*	2.9	5.9	1225	2450	420
29*	1.5	3.1	270	660	181
30*	0.9	1.8	600	1180	658
31	0.4	0.3	250	220	634
32	0.26	0.0	200	0	755

The discharge from the flow pipe was uniform and steady from 1 to 26 inclusive. In 27 a slight wavering of the contents of the column could be occasionally observed showing that this was just at the limit of steady flow. A slight further constriction of the clamp gave No. 28, which was perfectly regular and periodic, the discharge occurring at the uniform rate of 22 strokes per minute throughout the experiment. No. 29 was regular and periodic, discharging at the uniform rate of 15 strokes per minute. The discharge of No. 30 was periodic and very slow, the strokes were not counted. No. 31 was irregular, some strokes discharging water and others failing to do so. No. 32 was so adjusted that the supply of air was just sufficient to keep the water level in the flow pipe at its summit; thus the number of c.c of air 0.26 represents just the leakage for this particular experiment. Observe that in the periodic flows, the discharge computed per second can only be taken to represent a mean or average velocity of discharge during the whole experiment.

The periodic discharge appeared to be caused by an insufficient air supply. The bubbles appeared to interfere and jostle each other in the lower end of the flow pipe funnel and would accumulate there until the upward pressure was sufficient to drive out the contents of the pipe.

The bubbles which *filled* the bore were shaped like conical rifle balls while those which did not fill the bore were lenticular shaped revolutes whose equators were horizontal, the surfaces above the equators being much more curved than those below. These latter bubbles vibrated with great rapidity as they ascended the tube, this phenomenon was most observable in the periodic flows, where in the interval between the strokes the bubbles were nearly stationary in the tube. This vibration was evidently caused by the *leakage* around the bubbles. At no time was there any tendency of a bubble to break, but on the contrary the tendency was to unite when they approached each other closely.

In order to illustrate the results of these experiments more clearly, I have plotted the rate-flows to axes of A as abscissa and W as ordinate, Fig. 3. I have drawn approximately the locus of mean position, by a straight line between the points (0.25,0.0) and (4.0,10.0) and thence a parabola tangent thereto.

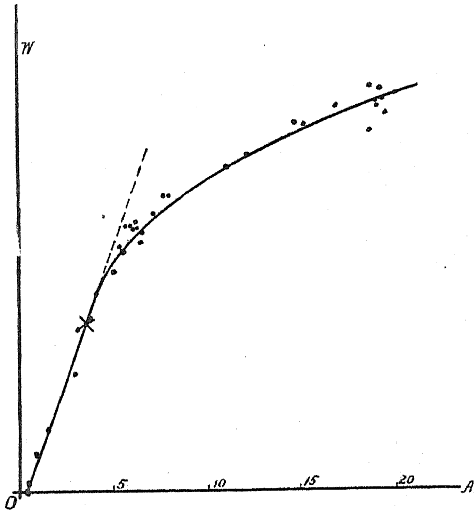


FIG. 3.

The equation to this parabola referred to tangent and horizontal diameter is

$$y^2 = 10x,$$

which transferred to A and W axes, gives A in terms of W.

Thus,

$$A = \tfrac{1}{4}(\tfrac{3}{2}W + 1)\Big\}_{0}^{21} + 0.114(W - 10)^2\Big\}_{10}^{21}.$$

The brace after each term with the high and low limits assigned, mere indicates the limits for W between which that term is to be used. The second term is thus not used below W=10 and the experiments were not carried beyond W=21. It would be interesting to know whether W would continue to increase with A or not.

The flow was periodic and regular up to the point marked × on the tangent, and from this point on it was steady and uniform.

The efficiency of the machine as a pump at the maximum discharge A=19; W=20; V=$97\frac{1}{2}$c.: gives, E=37%.

At an intermediate point, A=6.0; W=13.5; V=49c.; gives, E=62%.

At the limit of steady flow, A=3.1; W=8.2; V=$28\frac{1}{4}$c.; gives, E=$68\frac{60}{100}$%.

The best results seem to be gotten just at this stage; however, as a sand-lift velocity of discharge and not pump efficiency is to be desired.

I may in closing give a few other results which were obtained with the same pipe 20 cm. out of the water and 30 cm. in the water. Thus with the same numbering and notation as in table I, these results are tabulated in table II as follows.

The discharge from 1 to 9 was steady and uniform. No. 10 was the limit of steady flow. Nos. 11, 12, 13 and 14 were

TABLE II.

20 cm. out and 30 cm. in the water.

No.	A	W	Q_a	Q_w	T
1	20.8	17.8	1500	1280	72
2	19.9	16.5	1600	1335	81
3	21.1	15.5	1500	1100	71
4	17.5	15.3	875	765	50
5	18.3	15.3	550	460	30
6	19.4	15.2	1300	1020	67
7	16.2	14.4	1350	1200	83
8	14.5	14.2	1375	1350	95
9	13.7	13.6	1375	1360	100
10	14.2	13.6	1150	1100	81
11*	5.5	5.2	950	920	176
12*	6.2	4.8	1400	1090	224
13*	4.1	4.7	650	740	158
14*	3.8	4.1	700	850	205
15*	1.0	0.6	500	350	559
16	0.41	0.0	200	0	490

regular and periodic giving respectively 27, 24, 23, 22 strokes per minute. The strokes of No. 15 were not counted being very slow. In No. 16 the air delivery was so regulated as to just keep the pipe full to the top, and therefore 0.41 c.c.s. represents the leakage in sustaining a 20c. head in the pipe. Throughout the series the flows A and W are nearly equal, but that of W increases less rapidly than A.

The limit of periodic flow is much higher than in the first series. Unfortunately not enough determinations were made in the region W=10, to admit of drawing the graph of mean position for A and W, The efficiency as a pump is of course inferior to that for lower lifts.

A few other determinations were made for other lifts, as follows:

No.	h	d	Q_a	Q_w	T	S	E
1	25	35	500	450			0.64
2	25	35	1490	1320	333	22	0.60
3	15	45	300	550			0.61
4	30	30	375	350		20	0.94
5	12	48	200	510			0.63

Nos. 5 and 3 were steady flows, the others were all periodic, the number of strokes per minute are recorded under S, in the cases in which they were counted. The time was observed in only one case, that of number 2, whose efficiency may be computed. Thus for No. 2 we have $A=4.2$; $W=3.96$; $V=20.4$ and $h_v=0.00051V^2=0.2$. whence $E=0.60$.

The efficiency of the other cases has been computed, with h_v neglected, and tabulated under E. No. 4 is remarkable.

Finally, in closing, it may not be out of place to make a few remarks upon the subject of the loss of energy or that due to the leakage. While the experiments have not been sufficient in number, nor have they covered a wide enough range, to prevent any conclusions which may be drawn in regard to this matter being anything but premature, the results have suggested to me the following:

The loss of energy appears to be made up of two parts, that due to the sus taining of a statical head in the pipe above free level and an additional loss which is a function of the velocity of discharge.

The first part, which I shall call the potential loss, is dependent on the statical pressure intensity in the column; if there be no discharge this is h, if there be discharge this is $H=h+h_v$. This loss can be determined directly by regulating air supplies which will sustain free levels, in a pipe indefinitely extended upward, for different values of H, as was done in Nos. 32 and 16 of tables I and II respectively. Here the losses by leakage are equal to the air supplies. The results seem to indicate that this loss of work from leakage is directly proportional to H, the intensity of statical pressure in the pipe, or

$$l'=cH=c(h+h_v).$$

If l', be the loss equal to the flow of air which will sustain h without discharge, then

$$l'=l'_1\left(1+\frac{h_v}{h}\right).$$

Neglecting the loss due to frictional resistances in the pipe, the loss of kinetic energy or that loss which is a function of the velocity, is

$$l''=Ad-WH-cH,$$
$$=Ad-H(W+c).$$

Referring now to table I, (or better still to a curve of mean position) and computing from the data there given, the quantities V, h_v, l' and thence l'', we find that for the steady flows of uniform V, the series of values of l'' are proportional to those of h_v and therefore to V^2.

The loss l'' is therefore truly a loss of kinetic energy.

If we put
$$l''=\frac{k}{2g}V^2,$$
the original equation of energy becomes,
$$wAd=(wW+c)h+(wW+k)h_v+r.$$

In table I, we have
$$l'=l_1'(1+h_v/h)=12+h_v,$$
and within the limits of error of the experiment,
$$l''=100h_v=\frac{50}{g}V^2,$$
for uniform flow, $\therefore$ $k=0.051$.

For flows which are periodic, the law cannot be expected to hold when the values of V are gotten as above, since the discharge takes place in squirts of high velocity, followed by longer periods of rest. The actual velocity of mean flow must therefore be higher than the computed ones, which will bring the series for periodic flow under the law also.

In order to derive reliable conclusions from such experiments a large number of sets should be made upon pipes of different diameters, with a larger range of air flows and values of d and h.

Also different arrangements for introducing the air into the flow pipe should be employed, it is my opinion that introducing it into the side of the pipe, as did Mr. Harris, will do away with periodic flow altogether. I distinguish between periodic flow caused by accumulations of air in the lower end of the pipe, and intermittent flow as caused by the escape of air in the discharge.

I contemplate, if time be allowed, pushing these experiments further, and finally comparing its efficiency with that of the water motor described in the beginning of this paper.

EDITORIAL NOTE.

It has been decided to discontinue the publication of this Journal and its issue ceases with this number, which closes the first volume.

We close the first volume and cease the publication with considerable regret, yet with no small degree of satisfaction, believing as we do, that as a Journal of Elementary Mathematics it has accomplished fairly well the object which it had in view.

Had it done nothing more than to put into English words the papers of Bolyai and Lobatschewsky its life had been well lived. We believe that the time will yet come when the seed thus sown will bear its share of fruit in the advancement of sound geometrical teaching in America.

W. H. E.

SOLUTION OF EXERCISES.

ACKNOWLEDGMENTS.

C. B. Spencer 27; G. B. Halsted 27; J. C. Nagle 21; William Hoover 21; G. H. Harvill 23.

21.

A horizontal beam span a, resting on two supports at ends, is loaded so that the load per running foot varies as the square of the distance from one support. Find the tangent to the elastica at each end of the beam and the maximum deflection and that at the center. [*T. U. Taylor.*]

SOLUTION.

If u= the pressure at the distance of a unit from a support, at the distance z, the uz^2. The whole load is

$$\int_0^z uz^2dz=\tfrac{1}{3}uz^3,$$

and the center of gravity is

$$\bar{z}=\frac{\int_0^z uz^2dz.z}{\int_0^z uz^2dz}=\tfrac{3}{4}z,$$

from the support.

Let x be the distance of any vertical section of the load from the middle of the beam, and y= the deflection. Then E and I having the usual meanings, taking moments about the outer extremity of x

$$(1)\qquad \mathrm{EI}\frac{d^2y}{dx^2}=\frac{a}{2}\frac{ua^3}{24}-\tfrac{1}{4}(\tfrac{1}{2}a-x).\tfrac{1}{3}u(\tfrac{1}{2}a-x)^3.$$

Integrating once and noticing that when x=0, dy/dx=0,

$$(2)\qquad \mathrm{EI}\frac{dy}{dx}=\tfrac{1}{48}ua^4x+\tfrac{1}{60}u(\tfrac{1}{2}a-x)^5-\tfrac{1}{1920}ua^5.$$

When $x=\frac{1}{2}a$, (2) gives

$$\mathrm{EI}\ dy/dx=\tfrac{19}{1920}ua^5.$$

Equating the dexter of (2) to zero,

$$\tfrac{1}{5}(\tfrac{1}{2}a-x)^5-\tfrac{1}{4}a^4(\tfrac{1}{2}a-x)+\tfrac{19}{1920}a^5=0,$$

gives the maximum deflection.

Integrating (2) and noticing that when $x=\frac{1}{2}a$, $y=0$,

(3) $\mathrm{EI}y=\frac{1}{96}ua^4x^2-\frac{1}{360}u(\frac{1}{2}a-x)^6-\frac{1}{1920}ua^5x-\frac{3}{1280}ua^6.$

When $x=0$,

$$y=-\frac{11ua^6}{4608\mathrm{EI}},$$

the required deflection. [*William Hoover.*]

23.

Integrate

$$\frac{d^2u}{dx\,dy}=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}.$$

[*G. H. Harvill.*]

SOLUTION.

Put $1+y^2=b$, and then integrate with respect to x. Then

$$\frac{dx}{dy}=\frac{x}{b(b+x^2)^{\frac{1}{2}}}.$$

Put $1+x^2=a$, and integrate with respect to y. Then

$$u=x\int\frac{dy}{(1+y^2)(a+y^2)^{\frac{1}{2}}},$$

$$=x\int\frac{dz}{z(z^2-c)},$$

where $a-1=c$.

Whence finally

$$u=\frac{x}{2x^2}\log.\left(\frac{y^2+1}{y^2+x^2+1}\right)$$

$$=\log\frac{\sqrt{y^2+1}}{y^2+2}.$$

[*G. H. Harvill.*]

EXERCISES.

28.

The center of an equilateral triangle circumscribed to a parabola is the orthocenter of the points of contact.

[*Frank Morley.*]

29.

A triangle is inscribed in a parabola having its vertex at the point of contact of the tangent parallel to its base. In either of the segments made by its sides, another triangle is similarly inscribed. Show that the former triangle is eight times the latter. [*W. B. Richards.*]

30.

Two conics which pass each through the focus of the other have a common auxiliary circle. [*H. B. Newson.*]

31.

A parabola touches the double tangent of a fixed cardioid. Show that the eight points of intersection of the curves lie on two circles; that the circles meet on a fixed circle; and that the radical axis of the circles is parallel to the axis of the parabola.

[*Frank Morley.*]

32.

Integrate

$$\frac{\varphi \sin \varphi \, d\varphi}{(a+b \cos \varphi)^4}.$$

[*G. H. Harvill.*]

33.

Through a fixed point and the cusp of the cissoid of Diocles, three and only three circles can be passed cutting the cissoid at right angles; and these three points of intersection are collinear. [*H. B. Hall.*]

34.

Let I_1, I_2, I_3 be the escribed centers of a triangle A_1, A_2, A_3. Let the lines joining any point O to the vertices meet the opposite sides at P_1, P_2, P_3. Show that I_1P_1, I_2P_2, I_3P_3 are concurrent. [*Frank Morley.*]

35.

Required the volumes cut out of a cylinder with radius m, by another cylinder with radius n, the axes making with each other an angle $\theta=45^\circ$. [*G. H. Harvill.*]

36.

The locus of the middle-points of the intercepts on a pencil of lines by two given lines is an hyperbola whose asymptotes are parallel to the given lines. [*H. B. Newson.*]

37.

Given two con-nodal tri-nodal quartics, four conics can be passed through the three common double points and touching each of the quartics and their eight points of contact lie on another con-nodal tri-nodal quartic.

[*Annie L. Mackinnon.*]

CONTENTS.

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